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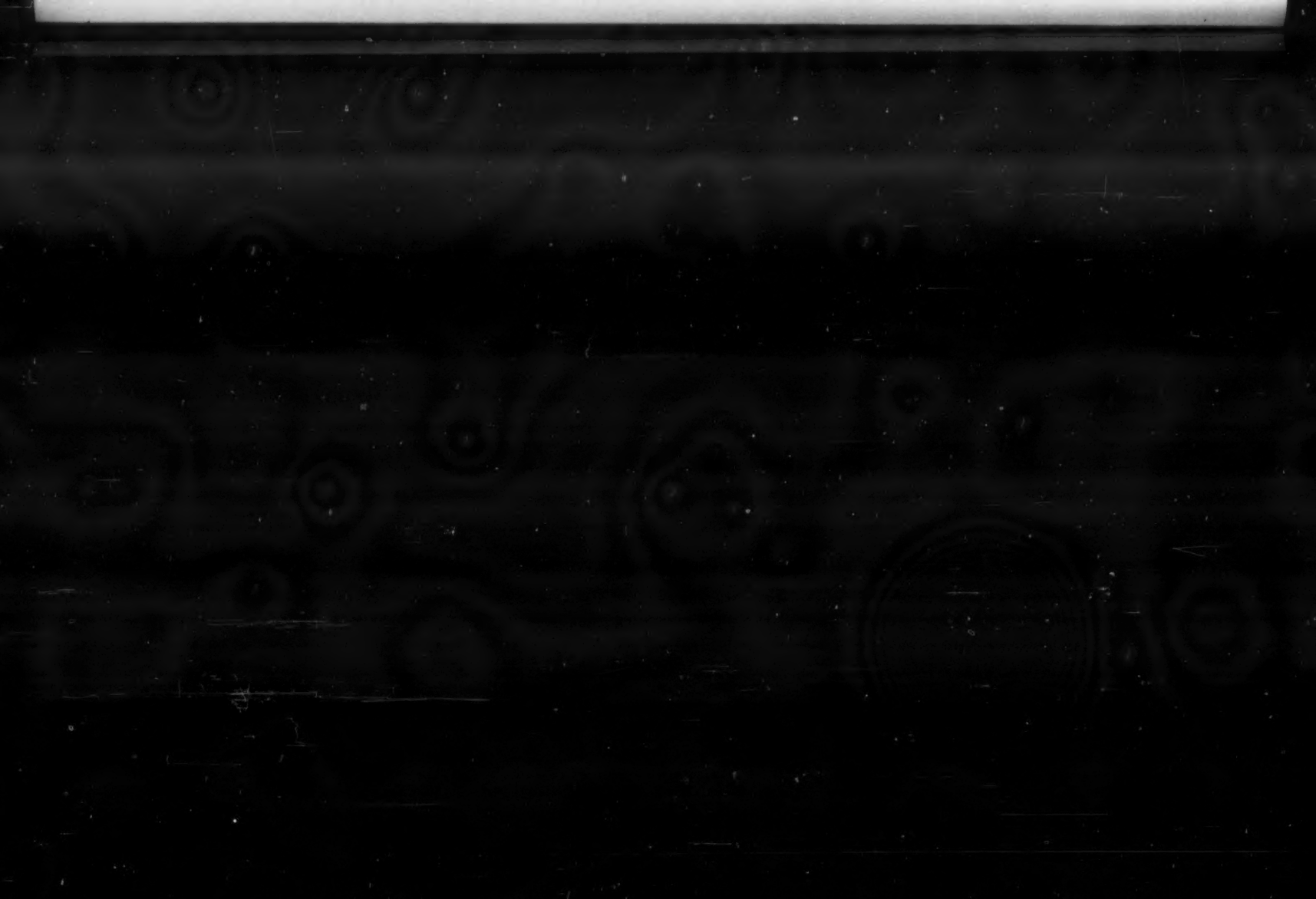

VARIATION, MULTIPLICITY, AND SEMICONTINUITY

LAMBERTO CESARI, Purdue University

The concept of total variation of a real function of one real variable is generally discussed in courses and books on real functions, or on integration. Nevertheless, the recent research on continuous varieties in Euclidean or metric spaces (curves, surfaces, *etc.*) have focused the attention on certain less-known aspects of the concept of total variation, whose extensions have an essential part in that research. In retrospect, these less-known facets of the familiar concept of total variation have become interesting. In dealing with these more elusive aspects of the concept of total variation we will come across a number of well-known properties, and we will list them without proof (See, *e.g.*, E. W. Hobson, *Theory of Functions of a Real Variable*, 1927; S. Saks, *Theory of the Integral*, 1937).

1. Jordan total variation. Given any single-valued, real, everywhere-finite function $f(u)$, $a \leqq u \leqq b$, and any subinterval $I = [\alpha, \beta]$ of $[a, b]$, by *variation* of f in I is meant, as usual, the number var $(f; I) = |f(\beta) - f(\alpha)|$ and by *oscillation* of f in I, the number osc $(f; I) = M - m$, where M, m denote the supremum and the infimum of f in I. Thus $0 \leqq \text{var } (f; I) \leqq \text{osc } (f; I) \leqq +\infty$. If $f(I)$ denotes the set of all real numbers $x = f(u)$ for some $u \in I$, *i.e.* $f(I) = [f(u),\ u \in I]$, and by diameter of a set we denote the supremum of the distances between pairs of points of the set, then we have again $M - m = \text{diam } f(I)$ and, therefore, osc $(f; I) = \text{diam } f(I)$.

If f is continuous in I, then osc $(f; I)$ is the difference $M - m$ between the maximum and the minimum of f in I. If f is monotone in I, *i.e.*, always nondecreasing or always nonincreasing, then var $(f; I) = \text{osc } (f; I)$.

Let $D = [a = u_0 < u_1 < \cdots < u_n = b]$ denote any finite subdivision of $[a, b]$ into subintervals $I_i = [u_{i-1}, u_i]$, $i = 1, \cdots, n$. Then the following two definitions of *total variation* appear natural:

$$V_1 = \sup_D \sum_{i=1}^{n} \text{var } (f; I_i), \qquad V_2 = \sup_D \sum_{i=1}^{n} \text{osc } (f; I_i),$$

where the supremum is taken in both with respect to all possible subdivisions D of $[a, b]$ as above. As a matter of fact, the following theorem holds:

(1.i) *For every function* $f(u)$, $a \leqq u \leqq b$, *we have* $V_1 = V_2$.

Proof. Obviously $V_1 \leqq V_2$. On the other hand, if $\epsilon > 0$ is any number, there is a subdivision D of $[a, b]$ in N parts such that, if $\Delta_i = \text{osc } (f; u_{i-1}, u_i)$, we have $\sigma = \sum_{i=1}^{N} \Delta_i > V_2 - \epsilon$ or $> \epsilon^{-1}$ according as $V_2 < +\infty$ or $V_2 = +\infty$. Now in each interval $[u_{i-1}, u_i]$ there are two points ξ_i', ξ_i'' with $|f(\xi_i'') - f(\xi_i')| > \Delta_i - \epsilon/N$ if $\Delta_i < +\infty$, $> \epsilon^{-1}$ if $\Delta_i = +\infty$. Now the points ξ_i', ξ_i'' divide each $[u_{i-1}, u_i]$ into one, or two, or three subintervals. If $D' = [a = v_0 < v_1 < \cdots < v_n = b]$, $N \leqq n \leqq 3N$, is the new subdivision of $[a, b]$, then the sum $\sum |f(v_j) - f(v_{j-1})|$ relative to the intervals $[v_{j-1}, v_j]$ contained in $[u_{i-1}, u_i]$ is $\geqq \Delta_i - \epsilon/N$ if $\Delta_i < +\infty$, $> \epsilon^{-1}$ if

$\Delta_i = +\infty$, and finally $\sum_{j=1}^{N} |f(v_j) - f(v_{j-1})| > \sigma - \epsilon$ or $> \epsilon^{-1}$. Thus $V_1 \geqq V_2$ and finally $V_1 = V_2$. Thereby (1.i) is proved.

As usual we shall define as *total variation* of $f(u)$ in $[a, b]$, the common value of V_1 and V_2, *i.e.*, $V = V(f; a, b) = V_1 = V_2$. Then $f(u)$ is said to be of bounded variation, or BV, in $[a, b]$ if $V < +\infty$. Obviously, if f is BV in $[a, b]$, then

$$\sum_{i=1}^{n} \text{var } (f; I_i) \leqq V < +\infty, \qquad \sum_{i=1}^{n} \text{osc } (f; I_i) \leqq V < +\infty,$$

for every subdivision D. Also f is bounded in $[a, b]$, since

$$|f(u)| = |f(u) - f(a) + f(a)| \leqq |f(a)| + V < +\infty.$$

If f is monotone in $[a, b]$ then $V = |f(b) - f(a)|$ and f is BV. If f has finitely-many relative maxima and minima, say at the points $a < \xi_1 < \cdots < \xi_k < b$, then $V = |f(a) - f(\xi_1)| + |f(\xi_1) - f(\xi_2)| + \cdots + |f(\xi_n) - f(b)|$ and f is BV. Thus for $f(u) = \sin u$, $0 \leqq u \leqq 2\pi$, we have $V = 4$. The two functions $f(u)$, $g(u)$, $0 \leqq u \leqq 1$, defined by $f(u) = u^2 \sin u^{-1}$, $g(u) = u \sin u^{-1}$, $0 < u \leqq 1$, $f(0) = g(0) = 0$, need a less elementary analysis. Both f, g are continuous, f is BV, and g is not BV in $[0, 1]$.

Remark 1. In the considerations above we have used the two interval functions $\phi_1(I) = \text{var } (f; I)$ and $\phi_2(I) = \text{osc } (f; I)$. Both are nonnegative, subadditive functions, *i.e.*, if we denote either one of these functions by $\phi(I)$, then for every I and every finite subdivision $(I_1, \cdots, I_m)$ of I into subintervals we have $0 \leqq \phi(I) \leqq \phi(I_1) + \cdots + \phi(I_m)$. Nevertheless $V(I) = V(f; I)$, *i.e.*, the total variation of f restricted to the interval $I = [\alpha, \beta]$, is an additive interval function, *i.e.*, $V(I) = V(I_1) + \cdots + V(I_m)$ [cf. (5.iii)].

Remark 2. If $f = g \pm h$, then $V(f) \leqq V(g) + V(h)$. If f, g are BV, then $f \pm g$, fg, $|f|$, $|g|$ are BV, and also f/g is BV provided $|g| \geqq m > 0$.

2. The Jordan total variation as a limit. Suppose that $f(u)$ is any continuous function of u in $[a, b]$. For every subdivision D let us denote by d the number $d = \max (u_i - u_{i-1})$, $i = 1, \cdots, n$; and by δ the number $\delta = \max \text{ osc } (f; u_{i-1}, u_i)$. Either $d \geqq 0$ or $\delta \geqq 0$ can be thought of as a *norm* of D. By the uniform continuity of $f(u)$ in $[a, b]$ it follows that $\delta \to 0$ as $d \to 0$, but the converse is not true, as it becomes clear when f is constant on some subinterval of $[a, b]$.

(2.i) *For every continuous function f we have*

$$V_1 = V_2 = \lim_{\delta \to 0} \sum_{i=1}^{n} \text{osc } (f; I_i) = \lim_{\delta \to 0} \sum_{i=1}^{n} \text{var } (f; I_i) \tag{2.1}$$

and the same holds as $d \to 0$.

Proof. The last part is a trivial consequence of (2.1) and of the remark above. Also the whole statement is trivial if $f(u)$ is a constant in $[a, b]$. Because of (1.i) we have

$$V_1 = V_2 \geqq \sum_{i=1}^{n} \operatorname{osc}(f; u_{i-1}, u_i) \geqq \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})|$$

for every subdivision D. Hence it is enough to prove that the last limit (2.1) exists and equals V_1. We may suppose f not constant in $[a, b]$. Given $\epsilon > 0$ let $D_0 = [a = u_0 < u_1 < \cdots < u_N = b]$ be any subdivision of $[a, b]$ such that

$$\sum_{i=1}^{N} |f(u_i) - f(u_{i-1})| > V - \epsilon \text{ or } 1/\epsilon \tag{2.2}$$

according as $V < +\infty$ or $V = +\infty$. The existence of D_0 follows from the definition of V. We may suppose that f is constant in no interval $[u_{i-1}, u_i]$, since in such a case, we could well suppress (or add) points u_i in D_0 without reducing the sum (2.2) until D_0 satisfies such a requirement. Let $\eta_0 > 0$ denote the minimum of the oscillations of f in the intervals $[u_{i-1}, u_i]$, $i = 1, \cdots, n$, and denote by $\eta > 0$ the number $\eta = \min [\eta_0/3, \epsilon/N]$. Now let $D = [a = v_0 < v_1 < \cdots < v_n = b]$ be any subdivision of $[a, b]$ with norm $\delta < \eta$. Since $\eta \leqq \eta_0/3$, no part $[v_{j-1}, v_j]$ may contain completely a part $[u_{i-1}, u_i]$. We may separate the intervals $[v_{j-1}, v_j]$ into two classes C', C'' by putting in C' those parts which contain in their interior points u_i, $i = 1, \cdots, N-1$, and in C'' all others (Fig. 1). There are at

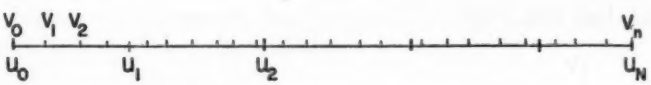

FIG. 1.

most $N-1$ intervals in C'. For each $[u_{i-1}, u_i]$ we have now a subdivision, say $u_{i-1} \leqq v_{j_s} \leqq v_{j_s+1} \leqq \cdots \leqq v_{j_s+t} \leqq u_i$, into parts, of which at most two are parts of intervals $[v_{j-1}, v_j]$ of C', and all others (at least one) are intervals of C''. We have

$$\begin{aligned} |f(u_i) - f(u_{i-1})| &\leqq |f(u_{i-1}) - f(v_{j_s})| + |f(v_{j_s+t}) - f(u_i)| + \sum |f(v_j) - f(v_{j-1})| \\ &\leqq 2\eta + \sum |f(v_j) - f(v_{j-1})|, \end{aligned} \tag{2.3}$$

where Σ ranges over all intervals $[v_{j-1}, v_j]$ of C'' in $[u_{i-1}, u_i]$. Obviously the first two terms in the second member of (2.3) do not exceed the oscillations of f in $[u_{i-1}, v_{j_s}]$, $[v_{j_s+t}, u_i]$ and hence each is less than η. Thus

$$\sum_{i=1}^{N} |f(u_i) - f(u_{i-1})| \leqq 2N\eta + \sum' |f(v_j) - f(v_{j-1})|,$$

where Σ' ranges over all $[v_{j-1}, v_j]$ of C'', and also

$$V_1 \geqq \sum_{i=1}^{n} |f(v_j) - f(v_{j-1})| \geqq \sum' |f(v_j) - f(v_{j-1})|$$

$$\geqq \sum_{i=1}^{N} |f(u_i) - f(u_{i-1})| - 2N\eta \geqq V_1 - \epsilon - 2\epsilon \text{ or } > 1/\epsilon$$

according as $V_1 < +\infty$ or $V_1 = +\infty$. This holds for all subdivisions D with $\delta < \eta$ and this proves that the last limit (2.1) is V_1. Thereby (1.i) is proved.

Remark 1. Statement (2.i) does not hold necessarily for functions f which are not continuous.

Remark 2. The reasoning used in the proof of (2.i) is a typical one. An analogous reasoning is used in the proof that the upper and lower Darboux integrals in the theory of Riemann integration are limits (and not just infimum and supremum of the respective sums). The same reasoning proves that the Jordan length of a continuous curve is a limit.

A theorem of C. Scheeffer concerning subadditive interval functions may be given which is general enough to comprehend these different situations. (See C. Scheeffer, Acta Math., vol. 5, 1884–85, p. 49, and extensions in L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. I, 1921–23, p. 39). A general theory of subadditive (or overadditive) interval- (and set-) functions has been given by S. Banach (Fund. Math., vol. 6, 1924, pp. 170–188). On the same subject see also the exposition of S. Kempisty, *Fonctions d'intervalle non-additive*, Actualités Sci. Ind. 824, 1939.

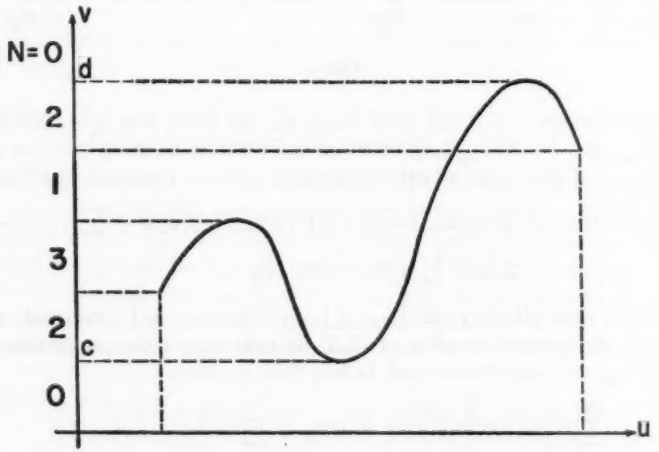

FIG. 2.

3. The multiplicity function. Again let $x = f(u)$, $a \leqq u \leqq b$, be any single-valued real continuous function, and let us denote by c and d its minimum and maximum in $[a, b]$. For every number ξ, $-\infty < \xi < +\infty$, let us denote by $N(\xi) = N(\xi; f, a, b)$ the number [zero, any positive integer, $+\infty$] of the solutions

of the equation $f(u)=\bar{x}$. This function $N(x)$, $-\infty<x<+\infty$, is usually denoted as the (*crude*) ***multiplicity*** function of $f(u)$. Obviously $N(x)$, $-\infty<x<+\infty$, is non-negative and $N(x)=0$ for all $x>d$ and all $x<c$ (Fig. 2). We will prove below that $N(x)$ is a Borel-measurable function and thus the number

$$V_2=\int_{-\infty}^{+\infty}N(x)dx \tag{3.1}$$

exists, $0\leqq V_2\leqq+\infty$. This number has been proposed by Banach as an alternative definition of total variation. Indeed the following theorem holds:

(3.i) (S. Banach). *For every continuous function f we have $V_1=V_2$.*

Proof. Let D_n denote the subdivision of $I=[a, b]$ into 2^n equal parts, and more precisely let I_{n1} denote the closed interval $[a, a+(b-a)/2^n]$, and I_{ni} all remaining intervals open at the left and closed at the right

$$I_{ni}=(a+(i-1)(b-a)/2^n, a+i(b-a)/2^n], \quad i=2, 3, \cdots, 2^n.$$

The function f maps each interval I_{ni} into a segment (closed or not) of the x-axis, namely, the segment, from c_i to d_i, where $c_i=\min f$, $d_i=\max f$ in I_{ni}. The characteristic function $\phi_{ni}(x)$ of the set $f(I_{ni})$ is therefore zero for $x>d_i$ and $x<c_i$, one for $c_i<x<d_i$, while it may be zero or one at the two end points. Thus $\phi_{ni}(x)$, $-\infty<x<+\infty$, is certainly B-measurable, and so is the function $\phi_n(x)=\sum_{i=1}^{2^n}\phi_{ni}(x)$, $-\infty<x<+\infty$. In addition,

$$\int_{-\infty}^{+\infty}\phi_n(x)dx=\sum_{i=1}^{2^n}\int_{-\infty}^{+\infty}\phi_{ni}(x)dx=\sum_{i=1}^{2^n}(d_i-c_i)=\sum_{i=1}^{2^n}\operatorname{osc}(f, I_{ni}) \tag{3.2}$$

and $\phi_n(x)\geqq 0$, $\phi_n(x)\leqq\phi_{n+1}(x)$ for all x and n. Thus the limit

$$\phi(x)=\lim_{n\to\infty}\phi_n(x), \qquad -\infty<x<+\infty,$$

exists everywhere in $(-\infty, +\infty)$ and $\phi(x)$ is also B-measurable. Finally, by a theorem of B. Levi (*cf.* Hobson, *loc. cit.*, Vol. I, p. 582), we have

$$\int_{-\infty}^{+\infty}\phi(x)dx=\lim_{n\to\infty}\int_{-\infty}^{+\infty}\phi_n(x)dx=V_2,$$

where the last equality follows from (3.2) and (2.i). All we have to prove now is that $\phi=N$. Indeed, if for a point $\bar{x}$ and an integer k (finite) we have $N(\bar{x})\geqq k$, there must be some k *distinct* points $\bar{u}$ with $f(\bar{u})=\bar{x}$, and these k points must belong to distinct intervals I_{ni} for all n large enough. Thus $\phi_{ni}(\bar{x})=+1$ for at least k different intervals I_{ni}, and hence $\phi(\bar{x})\geqq\phi_n(\bar{x})\geqq k$. Since k is any number $k\leqq N(\bar{x})$, we conclude that $\phi(\bar{x})\geqq N(\bar{x})$. Suppose now that $\phi(\bar{x})\geqq k$ for some $\bar{x}$ and integer k, then, since $\phi_n(\bar{x})\to\phi(\bar{x})$ and ϕ_n is necessarily an integer, we have $\phi_n(\bar{x})\geqq k$ for all large n, and hence $\bar{x}$ belongs to at least k distinct sets $f(I_{ni})$. Thus $f(\bar{u})=\bar{x}$ for at least k points $\bar{u}$ of k different intervals I_{ni}. Thus $N(\bar{x})\geqq k$,

and also $N(\bar{x}) \geqq \phi(\bar{x})$ since k is any integer $k \leqq \phi(\bar{x})$. This assures that $N(x) = \phi(x)$ for all x. Theorem (3.i) is proved. (See for another proof, S. Saks, *Theory of the Integral*, Warsawa, 1937, p. 280.)

Remark. If $I = [\alpha, \beta]$ and $[I_1, \cdots, I_m]$ is any finite subdivision of I into parts by means of the points $\alpha = u_0 < u_1 < \cdots < u_m = \beta$, then

$$N(x; f, I) = \sum_{i=1}^{m} N(x; f, I_i)$$

for all $x \neq f(u_j)$, $j = 0, 1, \cdots, m$. The proof is trivial.

4. The Jordan total variation as a Jordan length. The Jordan length of a curve is defined as the supremum of the elementary lengths of the inscribed polygonal lines. Now $x = f(u)$, $a \leqq u \leqq b$, can be thought of as a mapping of the interval $[a, b]$ of the u-axis E_1 into the x-axis E_1', hence as a "curve" C contained in the 1-dimensional space E_1'. Thus as u moves from a to b the point $x = f(u)$ moves from $f(a)$ to $f(b)$ but not necessarily monotonically. For instance, in Figure 3, $f(x)$ moves from $f(a)$ backwards to the minimum c of f, then after a number of forward and backward movements reaches the maximum d of f,

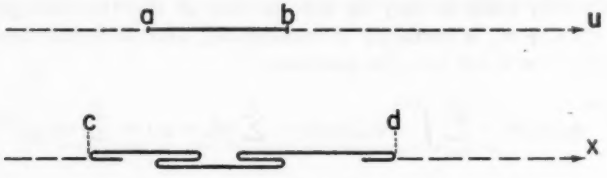

FIG. 3.

and ends in $f(b)$. We shall denote by V_4 the Jordan length of the wormlike curve C. Let us prove that $V_1 = V_4$. For every subdivision $D = [a = u_0 < u_1 < \cdots < u_n = b]$ of $[a, b]$ let $p_i = f(u_i)$ be the image of u_i on C, $i = 0, 1, \cdots, n$. Then the polygonal line $P = p_0 p_1 p_2 \cdots p_n$ inscribed in C, has sides $p_{i-1}p_i$ each of length equal to the distance of p_{i-1} and p_i, *i.e.*, $|f(u_i) - f(u_{i-1})|$. Thus the elementary length $l(P)$ of P is the sum of these numbers, and

$$V_4 = \sup l(P) = \sup \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})| = V_1.$$

5. Positive and negative total variations. If m denotes any real number we shall denote by m^+, m^- the numbers $m^+ = m$ if $m \geqq 0$, $m^+ = 0$ if $m \leqq 0$; $m^- = 0$ if $m \geqq 0$, $m^- = |m|$ if $m \leqq 0$. In other words $m^+ = 2^{-1}(|m| + m) \geqq 0$, $m^- = 2^{-1}(|m| - m) \geqq 0$, and hence $m^+ + m^- = |m|$, $m^+ - m^- = m$.

We shall now consider a single-valued real function $x = f(u)$, $a \leqq u \leqq b$. As usual we will denote by $D = [a = u_0 < u_1 < \cdots < u_n = b]$ any finite subdivision of $[a, b]$ and we will then denote by *positive* and *negative total variations*, V_+

$= V_+(f; a, b)$, $V_- = V_-(f; a, b)$ of f in $[a, b]$ the numbers $0 \leqq V_+$, $V_- \leqq +\infty$,

$$(5.1) \qquad V_+ = \sup_D \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^+, \qquad V_- = \sup_D \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^-.$$

Obviously, $0 \leqq V_+ \leqq V \leqq +\infty$, $0 \leqq V_- \leqq V \leqq +\infty$. Hence V_+, V_-, V are all finite if f is BV.

If we denote, for the sake of brevity, by $\sum$, $\sum_+$, $\sum_-$ the sums

$$\sum = \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})|, \qquad \sum_+ = \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^+,$$

$$\sum_- = \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^-,$$

we have

$$(5.2) \qquad \sum_+ + \sum_- = \sum, \qquad \sum_+ - \sum_- = f(b) - f(a),$$

and hence

$$(5.3) \qquad 2\sum_+ = \sum + f(b) - f(a), \qquad 2\sum_- = \sum - f(b) + f(a).$$

These relations show that when $\sum$ approaches its supremum V, then also $\sum_+$ and $\sum_-$ approach their suprema V_+, V_- respectively, and the converse is also true. As a corollary of (2.i) we have then

(5.i) *For every continuous function f we have*

$$V_+ = \lim_{\delta \to 0} \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^+, \qquad V_- = \lim_{\delta \to 0} \sum_{i=1}^{n} [f(u_i) - f(u_{i-1})]^-$$

and the same holds if $d \to 0$.

(5.ii) *For every function $f(u)$, $a \leqq u \leqq b$, we have $V_+ + V_- = V$, and for every BV function, we have also $V_+ - V_- = f(b) - f(a)$.*

Proof. This is a consequence of the formulas (5.2) and (5.3) and of the remark above.

(5.iii) *For every function $f(u)$, $u \in I = [a, b]$, and any finite subdivision $[I_1, \cdots, I_k]$ of I into subintervals we have*

$$V(I) = \sum_{j=1}^{k} V(I_j), \qquad V_+(I) = \sum_{j=1}^{k} V_+(I_j), \qquad V_-(I) = \sum_{j=1}^{k} V_-(I_j).$$

We omit the simple proof.

6. The functions $v(u)$, $v_+(u)$, $v_-(u)$. Given any function $f(u)$, $a \leqq u \leqq b$, we shall denote by $v(u)$, $v_+(u)$, $v_-(u)$, respectively, the total variation, the positive, and the negative total variations of f in $[a\ u]$:

$$v(u) = V(f; a, u), \qquad v_+(u) = V_+(f; a, u), \qquad v_-(u) = V_-(f; a, u).$$

Obviously, $v(x)$, $v_+(x)$, $v_-(x)$ are nonnegative, monotone nondecreasing functions, $0 \leqq v, v_+, v_- \leqq +\infty$, and $v(b) = V$, $v_+(b) = V_+$, $v_-(b) = V_-$. Thus v, v_+, v_- are everywhere finite if and only if f is BV. Also, $v(a) = v_+(a) = v_-(a) = 0$. As a corollary of (5.ii) we have now

(6.i) *For every BV function f we have $v_+(u) + v_-(u) = v(u)$, $v_+(u) - v_-(u) = f(u) - f(a)$ for all $a \leqq u \leqq b$.*

(6.ii) *A function f is BV if and only if $f = g - h$ is the difference of two monotone (everywhere-finite) functions.*

Statement (6.ii) is a corollary of (6.i) and of Remark 2 at the end of Section 1.

Remark 1. The decomposition mentioned in (6.i) is unique. Indeed, $v_+(u) = 2^{-1}[f(u) - f(a) + v(u)]$, $v_-(u) = 2^{-1}[-f(u) + f(a) + v(u)]$. A function $f(u)$, $a \leqq u \leqq b$, discontinuous at the point $\bar{u}$ is said to have a discontinuity of the first kind at $u = \bar{u}$ if both limits $f(\bar{u} - 0)$, $f(\bar{u} + 0)$ exist and are finite (and are not both equal to $f(\bar{u})$). Monotone (everywhere-finite) functions have only discontinuities of the first kind. Consequently, BV-functions also have only discontinuities of the first kind.

(6.iii) *If $f(u)$, $a \leqq u \leqq b$, is BV in $[a, b]$ and is continuous at $u = u_0$, then also $v(u)$, $v_+(u)$, $v_-(u)$ are continuous at $u = u_0$.*

This property is usually given in all expositions.

(6.iv) *A continuous function f is BV if and only if it is the difference of two monotone nondecreasing continuous functions.*

A consequence of (6.i) and (6.iii).

As is well known, a function f is said to be AC in $[a, b]$ if, given $\epsilon > 0$, there is a $\delta > 0$, such that, for every finite set of nonoverlapping intervals $[\alpha_i, \beta_i]$, $i = 1, \cdots, n$, with $\sum(\beta_i - \alpha_i) < \delta$, we have $\sum|f(\beta_i) - f(\alpha_i)| < \epsilon$. The following statements too are well known, and easily proved: (α) An AC function $f(u)$, $a \leqq u \leqq b$, is continuous and BV. (β) A function $f(u)$, $a \leqq u \leqq b$, is AC if and only if the function $v(u)$ is AC.

Remark. Examples of continuous BV functions which are not AC are usually given in all expositions. One of them is the nondecreasing continuous function $\phi(u)$, $0 \leqq u \leqq 1$, $\phi(0) = 0$, $\phi(1) = 1$, which is constant on each of the complementary intervals of the ternary Cantor set (Hobson, *loc. cit.*, Vol. I, p. 123, p. 368).

7. Lower semicontinuity of the total variation. We come now to one of the most important properties of the total variation: its lower semicontinuity. By this we mean that, if $f(u)$, $a \leqq u \leqq b$, is any function and $f_n(u)$, $a \leqq u \leqq b$, $n = 1, 2, \cdots$, a sequence of functions approaching $f(u)$, *i.e.*, $f(u) = \lim f_n(u)$ as

$n \to \infty$ for all u, if V, V_n are the total variations of f, f_n in $[a, b]$, then (7.i, 7.ii)

$$V \leqq \liminf_{n\to\infty} V_n. \tag{7.1}$$

To understand this property, let us suppose first that f is continuous and has finitely-many maxima and minima at points $a < \xi_1 < \cdots < \xi_k < b$ (Fig. 4). Let S be the sum $|f(a)-f(\xi_1)|+|f(\xi_1)-f(\xi_2)|+\cdots+|f(\xi_n)-f(b)|$ and S_n, the analogous sum for f_n and the *same points* $\xi_1, \cdots, \xi_k$. Since $f(a)=\lim f_n(a)$, $f(\xi_i)=\lim f_n(\xi_i)$, $f(b)=\lim f_n(b)$ as $n\to\infty$, $i=1, \cdots, k$, we have $S=\lim S_n$ as $n\to\infty$. By $V=S$, $S=\lim S_n$, $S_n \leqq V_n$, we conclude that (7.1) holds. Also, we understand that because of small oscillations of each of the functions f_n, in the $k+1$ intervals $[a, \xi_1], \cdots, [\xi_k, b]$ (oscillations not affecting the convergence of the sequence $f_n(u)$), it may occur that V_n is very large in comparison with V. That this situation actually may occur is shown by the examples below. On the other hand, the property (7.1) is very general as the statements (7.i), (7.ii) below show.

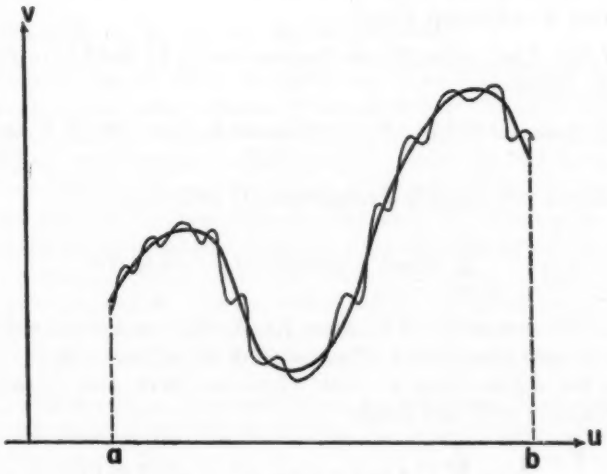

FIG. 4.

Example 1. Suppose $x=f(u)=0$, $0 \leqq u \leqq 1$, and $x=f_n(u)$, $0 \leqq u \leqq 1$, is represented by the polygonal lines OBA, OC_1MC_2A, $OD_1N_1D_2MD_3N_2D_4A, \cdots$ of Figure 5, *i.e.*, $f_n(u)=u-2^{-n}i$ if $w^{-n}i \leqq u \leqq 2^{-n}i+2^{-n-1}$, $f_n(u)=2^{-n}(i+1)-u$ if $2^{-n}i+2^{-n-1} \leqq u \leqq 2^{-n}(i+1)$, $i=0, 1, \cdots, 2^n-1$, $n=1, 2, \cdots$. Then $f=\lim f_n$ uniformly in $[0, 1]$, $V=0$, $V_n=1$, $n=1, 2, \cdots$, and $0=V<\lim V_n=1$.

Example 2. Let $f(u)=0$, $f_n(u)=n^{-1}\sin n^s u$, $0 \leqq u \leqq 2\pi$, for some real s and those $n \geq 0$ for which n^s is an integer. Then $f=\lim f_n$ uniformly in $[0, 2\pi]$, $V=0$, $V_n=4n^{s-1}$. Thus for $s=1$, we have $0=V<\lim V_n=4$; for $s=2$, we have

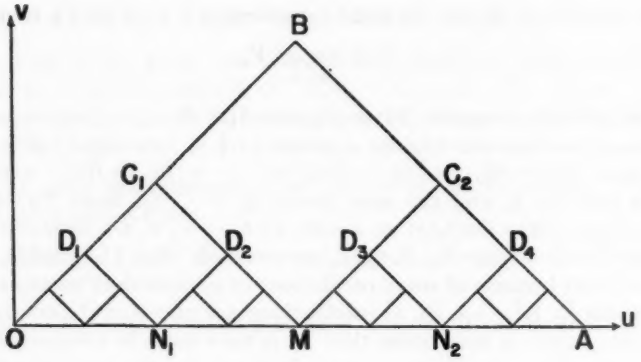

FIG. 5.

$0=V<\lim\ V_n=+\infty$; for $s=2^{-1}$ if $n=1, 4, 9, \cdots$, $s=1$ otherwise, we have $0=V=\liminf V_n<\limsup V_n=1$.

(7.i) *If $f(u)$, $f_n(u)$, $a\leqq u\leqq b$, are functions in $[a, b]$ and $f_n(u)\to f(u)$ as $n\to\infty$ for all $u\in[a, b]$, then*

$$(7.2)\quad V(f)\leqq \liminf_{n\to\infty} V(f_n),\quad V_+(f)\leqq \liminf_{n\to\infty} V_+(f_n),\quad V_-(f)\leqq \liminf_{n\to\infty} V_-(f_n).$$

Proof. Given $\epsilon>0$ there is a subdivision D such that

$$\sum_{i=1}^{N} |f(u_i)-f(u_{i-1})| > V-\epsilon \text{ or } 1/\epsilon$$

according as $V<+\infty$ or $V=+\infty$. Since $f_n(u_i)\to f(u_i)$ as $n\to\infty$, $i=0, 1, \cdots, N$, there is an n_i such that $|f_n(u_i)-f(u_i)|<\epsilon/2N$ for $n\geqq n_i$, $i=0, 1, \cdots, N$, and hence also for $n\geqq n_0=\max n_i$. For $n\geqq n_0$ we have also $|f_n(u_i)-f_n(u_{i-1})|>|f(u_i)-f(u_{i-1})|-\epsilon/N$ and finally

$$\left.\begin{matrix} V-\epsilon \\ 1/\epsilon \end{matrix}\right\} < \sum_{i=1}^{N} [|f_n(u_i)-f_n(u_{i-1})|+\epsilon/N] \leqq V(f_n)+\epsilon,$$

for all $n\geqq n_0$. This implies (7.2). Analogous proof holds for V_+ and V_-.

A variant of (7.i) is the following, for which we omit the proof.

If $f(u)$, $a\leqq u\leqq b$, is a continuous function, if $f_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, are functions with $f_n(u)\to f(u)$ as $n\to\infty$ at least for all u of a set which is everywhere dense in $[a, b]$, then relations (7.2) hold.

Remark. In neither of the theorems above uniform convergence is required, and in the first one the continuity of the functions is not required.

8. On maxima and minima of continuous functions. Before we discuss further questions concerning the multiplicity function N, we mention some known results on maxima and minima of continuous functions, which are of interest in themselves.

A point u_0 is said to be a point of weak [strict] relative maximum for a function $f(u)$, $a \leqq u \leqq b$, if (i) $a < u_0 < b$; and (ii) there is a neighborhood U of u_0 where $f(u) \leqq f(u_0)$ for all $u \in U$ [$f(u) < f(u_0)$ for all $u \in U$, $u \neq u_0$]. Analogous definitions hold for weak and strict relative minima. Let E_w, e_w, E_s, e_s be the sets of all points $u \in (a, b)$ of weak maximum, weak minimum, strict maximum, strict minimum, respectively. Let M_w, m_w, M_s, m_s be the sets of all values x taken by $f(u)$ at the points u of E_w, e_w, E_s, e_s respectively; that is,

$$M_w \equiv [f(u),\ u \in E_w] = f(E_w), \qquad m_w \equiv [f(u),\ u \in e_w] = f(e_w),$$

$$M_w \equiv [f(u),\ u \in E_s] = f(E_s), \qquad m_s \equiv [f(u),\ u \in e_s] = f(e_s),$$

$$E_s \subset E_w, \qquad M_s \subset M_w, \qquad e_s \subset e_w, \qquad m_s \subset m_w.$$

(8.1) *The sets E_s, e_s, M_s, m_s, M_w, m_w are countable.*

Proof. Consider the set A_n of all points $u_0 \in E_s$ such that $f(u) < f(u_0)$ for all $u_0 - 1/n < u < u_0 + 1/n$, $u \neq u_0$. Obviously $E_s = A_1 + A_2 + \cdots$. Let us prove that, if two points u_1, $u_2 \in A_n$, then $|u_1 - u_2| \geqq 1/n$. Indeed, in the contrary case, we have $f(u_1) < f(u_2)$, $f(u_2) < f(u_1)$, a contradiction. Thus each set A_n is finite and E_s is countable. Analogous proof holds for e_s. The sets M_s, m_s are also countable since they are the sets of values taken by $f(u)$ on E_s, e_s respectively.

Consider the set B_n of all points $u_0 \in E_w$ such that $f(u) \leqq f(u_0)$ for all $u_0 - 1/n < u < u_0 + 1/n$ and $C_n = f(B_n)$. Obviously $M_w = C_1 + C_2 + \cdots$. Let us prove that if two numbers x_1, $x_2 \in C_n$, $x_1 \neq x_2$, and $x_1 = f(u_1)$, $x_2 = f(u_2)$, u_1, $u_2 \in B_n$, then $|u_1 - u_2| \geqq 1/n$. Indeed, in the contrary case, we would have $x_1 \geqq x_2$, $x_1 \leqq x_2$, and hence $x_1 = x_2$, a contradiction. Thus each set C_n is finite and M_w is countable.

Remark. The sets E_w, e_w may well be uncountable since, if f is constant in an interval $[\alpha, \beta]$, then $[\alpha, \beta] \subset E_w$, $[\alpha, \beta] \subset e_w$. For some of the statements above cf. S. Saks, loc. cit., p. 261.

9. The multiplicity functions N_c, N_+, N_- and their properties of lower semicontinuity. The number $N(\bar{x})$ answers the question of how many times the continuous function $f(u)$ takes the value $\bar{x}$ in $[a, b]$, or, geometrically, how many points the curve $x = f(u)$, $a < u < b$, has in common with the straight line $x = \bar{x}$. Analogously we may ask how many times the same curve "crosses" the straight line $x = \bar{x}$, or how many times it does so from below to above, or from above to below. The answer is given by the functions $N_c(x)$, $N_+(x)$, $N_-(x)$ defined below, the first of which coincides with $N(x)$ for all but countably many points, and all related to the corresponding total variations V, V_+, V_- of $f(u)$ in $[a, b]$ by the Banach-type relations

$$V = \int_{-\infty}^{+\infty} N(x)dx = \int_{-\infty}^{+\infty} N_c(x)dx, \quad V_+ = \int_{-\infty}^{+\infty} N_+(x)dx, \quad V_- = \int_{-\infty}^{+\infty} N_-(x)dx.$$

The functions $N_c(x)$, $N_+(x)$, $N_-(x)$ are called the *corrected multiplicity*, the *positive* and *negative multiplicity* of $f(x)$ in $[a, b]$. The functions $N_c(x)$, $N_+(x)$, $N_-(x)$ have properties of lower semicontinuity, both with respect to x and to f, which will be stated below. These properties are analogous to the ones proved above for V, V_+, V_- (8.i). The crude multiplicity function $N(x)$ is not lower semicontinuous as examples immediately show.

For any interval $I=[\alpha, \beta]\subset[a, b]$ with $\alpha<\beta$, let $\phi(x; I, f)$ be the function defined by

$$\phi(x) = \phi(x; I, f) = \begin{cases} +1 & \text{if } f(\alpha) < x < f(\beta), \\ -1 & \text{if } f(\alpha) > x > f(\beta), \\ 0 & \text{otherwise.} \end{cases} \qquad -\infty < x < +\infty,$$

The function $\phi(x; I, f)$ has the following property of addition:

If $\alpha=u_0<u_1< \cdots <u_m=\beta$ is any finite subdivision of $I=[\alpha, \beta]$ into the m intervals $I_i=[u_{i-1}, u_i]$, $i=1, \cdots, m$, and $f(u_i)\neq x$, $i=0, 1, \cdots, m$. then

$$\phi(x; I, f) = \sum_{i=1}^{m} \phi(x; I_i, f). \tag{9.1}$$

Indeed, if $d_0=f(\alpha)-\bar{x}$, $d_m=f(\beta)-\bar{x}$ have opposite signs, say $d_0<0<d_1$, and $d_i=f(u_i)-\bar{x}$, $i=0, 1, \cdots, m$, then the sequence $d_0, d_1, \cdots, d_m$ is made up of numbers all $\neq 0$, and it must have at least one variation, more precisely, one variation from $-$ to $+$ more than variations from $+$ to $-$. The other cases can be dealt with analogously.

Let us now denote, as usual, by $\phi^+(x)$, $\phi^-(x)$ the functions

$$\phi^+(x) = \phi^+(x; I, f) = 2^{-1}[|\phi(x)| + \phi(x)] = \begin{cases} +1 & \text{if } f(\alpha) < x < f(\beta), \\ 0 & \text{otherwise;} \end{cases}$$

$$\phi^-(x) = \phi^-(x; I, f) = 2^{-1}[|\phi(x)| - \phi(x)] = \begin{cases} +1 & \text{if } f(\alpha) > x > f(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously we have $0\leqq\phi^+, \phi^-\leqq|\phi|$, $\phi^++\phi^-=|\phi|$, $\phi^+-\phi^-=\phi$ for all $-\infty<x<+\infty$. Let us observe that (9.1) implies

$$|\phi(x; I, f)| \leqq \sum_{i=1}^{m} |\phi(x; I_i, f)|$$

and, by using the definitions above, we have also

$$\phi^+(x; I, f) \leqq \sum_{i=1}^{m} \phi^+(x; I_i, f), \qquad \phi^-(x; I, f) \leqq \sum_{i=1}^{m} \phi^-(x; I_i, f). \tag{9.2}$$

Let $S=[I]$ be any finite system of nonoverlapping intervals $I=[\alpha, \beta]$ in $[a, b]$ and denote by $N_e(x)=N_e(x; f)$, $N_+(x)=N_+(x; f)$, $N_-(x)=N_-(x; f)$ the functions defined by

$$N_e(x) = \sup_S \sum_{I\in S} |\phi(x; f, I)|, \qquad N_+(x) = \sup_S \sum_{I\in S} \phi^+(x; f, I),$$

$$N_-(x) = \sup_S \sum_{I\in S} \phi^-(x; f, I).$$

Obviously, $0\leqq N_e, N_+, N_-\leqq +\infty$, $N_+\leqq N$, $N_-\leqq N$ for all $-\infty<x<+\infty$.

(9.i) *For every continuous function* $x=f(u)$, $a\leqq u\leqq b$, *we have* $0\leqq N_e(x)\leqq N(x)$ *for all* $-\infty<x<+\infty$, *and also* $N_e(x)=N(x)$ *for all but countably many* x.

(9.ii) *For every continuous function* $x=f(u)$, $a\leqq u\leqq b$, *we have* $N_e(x)=N_+(x)+N_-(x)$ *for all* $-\infty<x<+\infty$.

Proof of (9.i) *and* (9.ii). If $k\leqq N_e(\bar{x})$ for some $\bar{x}$ and integer k, then $f(u)-\bar{x}$ changes sign in at least k nonoverlapping intervals I, and thus $\bar{x}$ is taken in interior points of at least k nonoverlapping intervals, hence in at least k distinct points. Thus $k\leqq N(\bar{x})$ and finally $N_e(\bar{x})\leqq N(\bar{x})$. Suppose that $\bar{x}$ is any real number with $\bar{x}\neq f(a)$, $\bar{x}\neq f(b)$, $\bar{x}\notin M_w+m_w$, and suppose $k\leqq N(\bar{x})$ for some integer k. There must be at least k distinct points $\bar{u}$ in $[a, b]$ with $f(\bar{u})=\bar{x}$. Consider k disjoint open intervals U each containing one point $\bar{x}$. Since $\bar{x}\neq f(a)$, $\bar{x}\neq f(b)$, $\bar{x}\in M_w+m_w$, the difference $f(u)-\bar{x}<f(u)-f(\bar{u})$ changes sign in each of these intervals, otherwise $\bar{x}$ would belong to M_w+m_w. Therefore, in each U there is a closed subinterval $I=[\alpha, \beta]$ such that $f(u)-\bar{x}$ has values of opposite signs at α and β. Thus $|\phi(\bar{x}; f, I)|=1$ and, since the k intervals I are not overlapping, we have also $N_e(\bar{x})\leqq k$. Since k is any integer $k\leqq N(\bar{x})$ we conclude that $N(\bar{x})\leqq N_e(\bar{x})$ and finally, $N(\bar{x})=N_e(\bar{x})$ for all but countably many $\bar{x}$. Thus (9.i) is proved.

For any $\bar{x}$ let k be any integer with $k\leqq N_e(\bar{x})$. Then there are at least k nonoverlapping intervals $I=[\alpha, \beta]$ such that $f(\alpha)<\bar{x}<f(\beta)$ or $f(\alpha)>\bar{x}>f(\beta)$, that is, either $\phi(\bar{x}; I, f)=+1$ or $\phi(\bar{x}; I, f)=-1$. If k_1, k_2 are the numbers of intervals I of the first and of the second type we have $k=k_1+k_2\leqq N_+(\bar{x})+N_-(\bar{x})$. Since this holds for every $k\leqq N_e(\bar{x})$ we conclude that $N_e(\bar{x})\leqq N_+(\bar{x})+N_-(\bar{x})$. Suppose now that $k_1\leqq N_+(\bar{x})$, $k_2\leqq N_-(\bar{x})$ for some two integers k_1, k_2. There must be k_1 nonoverlapping intervals, say I_+, with $\phi^+(\bar{x}; I_+, f)=+1$; and k_2 nonoverlapping intervals, say I_-, with $\phi^-(\bar{x}; I_-, f)=+1$. On the other hand, it may well occur that intervals I_- are partially overlapping with intervals I_+ and vice versa. Let D be the finite subdivision of $[a, b]$ into parts which is obtained by using all end points of the intervals I_+, I_- as points of subdivision. By (9.2) we deduce that in each interval I_+ there must be at least one part, say J_+, with $\phi^+(\bar{x}; J_+, f)=+1$; and in each interval I_- there must be at least one part, say J_-, with $\phi^-(\bar{x}; J_-, f)=+1$. Thus there are in D at least k_1 intervals J_+ and at least k_2 intervals J_-, and these intervals are all distinct and nonoverlapping. Obviously, $|\phi(\bar{x}; J, f)|=+1$ for all intervals J_+ and J_-; hence $k_1+k_2\leqq N_e(\bar{x})$. We conclude that $N_+(\bar{x})+N_-(\bar{x})\leqq N_e(\bar{x})$ and, finally, $N_+(\bar{x})+N_-(\bar{x})=N_e(\bar{x})$.

(9.iii) *For every continuous function* $f(u)$, $a \leqq u \leqq b$, *the functions* $N_e(x)$, $N_+(x)$, $N_-(x)$ *are lower semicontinuous in* $(-\infty, +\infty)$; *that is, for every* x_0, *we have*

$$(9.3)\quad N_e(x_0) \leqq \liminf_{x \to x_0} N_e(x), \quad N_+(x_0) \leqq \liminf_{x \to x_0} N_+(x), \quad N_-(x_0) \leqq \liminf_{x \to x_0} N_-(x).$$

Proof. If $k \leqq N_e(x_0)$ for some integer k, then there are k disjoint intervals $I = [\alpha, \beta]$ with $f(\alpha) < x_0 < f(\beta)$ or $f(\alpha) > x_0 > f(\beta)$. Thus if $m > 0$ is the minimum of the $2k$ differences $|f(\alpha) - x_0|$, $|f(\beta) - x_0|$, we have $f(\alpha) < x < f(\beta)$ or $f(\alpha) > x > f(\beta)$ for every x with $|x - x_0| < m$ and the same k disjoint intervals I. This implies that $N_e(x) \geqq k$ for all $x_0 - m < x < x_0 + m$. Thus if $N_e(x_0) < +\infty$ and we take $k = N_e(x_0)$, then we have $N_e(x) \geqq N_e(x_0)$ for all x of a neighborhood of x_0; if $N_e(x_0) = +\infty$, then for every integer k we have $N_e(x) \geqq k$ in some neighborhood of x_0. This certainly implies the first of (9.3). Analogous reasoning holds for N_+ and N_-.

(9.iv) *If* $f(u)$, $f_n(u)$, $a \leqq u \leqq b$, $n = 1, 2, \cdots$, *are continuous functions and* $f_n(u) \to f(u)$ *everywhere in* $[a, b]$, *or at least for all* u *of a set which is everywhere dense in* $[a, b]$, *then*

$$N_e(x, f) \leqq \liminf_{n \to \infty} N_e(x; f_n), \qquad N_+(x, f) \leqq \liminf_{n \to \infty} N_+(x; f_n),$$

$$N_-(x; f) \leqq \liminf_{n \to \infty} N_-(x; f_n).$$

The proof is analogous to the one for (9.iii).

Remark 1. Statements (9.iii) and (9.iv) assure that the functions N_e, N_+, N_- are lower semicontinuous both with respect to x and with respect to f separately. Actually the functions N_e, N_+, N_- are lower semicontinuous with respect to the pair (x, f). That is, if $x \to x_0$, $f_n \to f$, then $N_e(x_0, f) \leqq \liminf N_e(x, f_n)$ as $x \to x_0$, $f_n \to f$. We omit the proof.

Remark 2. If for $-\infty < \bar{x} < +\infty$, one of $N_e(\bar{x})$, $N_+(\bar{x})$, $N_-(\bar{x})$ is finite, then all three are finite, and, if $\bar{x} \neq f(a)$, $\bar{x} \neq f(b)$, then $N_+(\bar{x}) - N_-(\bar{x}) = \phi(\bar{x}; f, [a, b])$. Also, $N_e(\bar{x})$ is attained for a subdivision of $[a, b]$ if and only if $N_+(\bar{x})$ and $N_-(\bar{x})$ are attained for the same subdivision. We omit the proofs.

10. A necessary and sufficient condition for absolute continuity.

(10.i) *A continuous BV function* $x = f(u)$, $a \leqq u \leqq b$, *is AC if and only if every set* $E \subset [a, b]$ *of measure zero is mapped by* f *into a set* $f(E)$ *of measure zero.*

This statement is usually given in all expositions.

11. The theorem of Lebesgue and its corollaries.

(11.i) (H. Lebesgue) *Every real monotone nondecreasing function* $F(u)$, $a \leqq u \leqq b$, *has a finite derivative* $0 \leqq F'(u) < +\infty$ *almost everywhere,* $F'(u)$ *is L-integrable in* $[a, b]$, *and* $F(b) - F(a) \geqq \int_a^b F'(u)\,du$. *The equality sign holds if and only if* F *is AC in* $[a, b]$.

This statement is usually given in the expositions together with the theorem (11.ii) below. The proof of (11.i) is based on Vitali's covering theorem.

(11.ii) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV *and* $v(u)$ *is the corresponding total variation in* $[a, u]$, *then the derivatives* f', v' *exist almost everywhere,* $|f'(u)| = v'(u)$ *almost everywhere* (a.e.) *in* $[a, b]$.

(11.iii) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV, *then* $f'(u)$ *exists and is finite* a.e. *in* $[a, b]$ *and* $\int_a^b |f'(u)|\,du \leqq V(f; a, b)$. *The equality sign holds if and only if* $f(u)$ *is* AC.

(11.iv) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV, *then the derivatives* $v'(u)$, $v'_+(u)$, $v'_-(u)$ *exist and are finite* a.e. *in* $[a, b]$, *and* (a) $v' = |f'|$, $v' = v'_+ + v'_-$, $f' = v'_+ - v'_-$ a.e.; (b) *either* $v'_+ = f'$, $v'_- = 0$, *or* $v'_+ = 0$, $v'_- = |f'|$ a.e.; (c) $\int_a^b v'\,du \leqq V$, $\int_a^b v'_+\,du \leqq V_+$, $\int_a^b v'_-\,du \leqq V_-$.

Proof of (11.iii) *and* (11.iv). The derivatives v', v'_+, v'_- exist a.e. by virtue of (11.i) and then f' exists a.e. and $f' = v'_+ - v'_-$ by (6.i). Also $|f'| = v' = v'_+ + v'_-$ by (6.i) and (11.ii). Thus (a) is proved. As a consequence we have $v'_+ = 2^{-1}[|f'| + f']$, $v'_- = 2^{-1}[|f'| - f']$, and thus $v'_+ = f'$, $v'_- = 0$ if $f' > 0$; $v'_+ = 0$, $v'_- = |f'|$ if $f' < 0$; $v'_+ = v'_- = 0$ if $f' = 0$. Thus (b) is proved. Relations (c) follow from (11.i). Finally $\int_a^b |f'|\,du = \int_a^b v'\,du \leqq v(b) - v(a) = V(f; a, b)$. Now the equality sign holds if and only if $v(u)$ is AC and, by statement (β) of Section 6, we know that this occurs if and only if $f(u)$ is AC. Thus (11.iii) and (11.iv) are proved.

Remark. Part (b) of (11.iv) is a first instance of a property of separation of v_+, v_- of which also curves and surfaces show very important extensions. Less elementary interpretations of (11.iv.b) are given in the following lines.

Let $f(u)$ be continuous and BV in $[a, b]$, and $0 < V = v(b) < +\infty$. Then $s = v(u)$ is constant on a subinterval of $[a, b]$ if and only if $f(u)$ is constant there and thus the function $F(s)$, $s \in J = [0, V]$, defined by $F(s) = f(u)$ whenever $s = v(u)$, is continuous in J, and $F(0) = f(a)$, $F(V) = f(b)$. If $w(s)$, $w_+(s)$, $w_-(s)$, $s \in J$, denote the total variations of F in $[0, s]$ (Sec. 6), we have $F(s) - F(0) = w_+(s) - w_-(s)$, $w_+(s) + w_-(s) = w(s) \equiv s$ for all $s \in J$. Also, a.e. in $[0, V]$, we have $F' = w'_+ - w'_-$, $w'_+ + w'_- = w' \equiv 1$ and, by (11.iv.b), either $w'_+ = 1$, $w'_- = 0$, or $w'_+ = 0$, $w'_- = 1$ a.e. in J. Since $0 \leqq w_+(s') - w_+(s)$, $w_-(s') - w_-(s) \leqq s' - s$ for all $0 \leqq s < s' \leqq V$, we conclude that $w_+(s)$, $w_-(s)$ are Lipschitzian with constant 1 in J, hence AC. Thus $w_+(s) = \int_0^s w'_+\,ds$, $w_-(s) = \int_0^s w'_-\,ds$. If $\mathfrak{B}$ denotes the ring of all B-measurable sets $\tilde{M} \subset J$, then for every set $\tilde{M} \in \mathfrak{B}$ we shall denote by $w(\tilde{M}) = |M|$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ the integrals taken on $\tilde{M}$ of the following functions $w' \equiv 1$, w'_+, w'_-, F'. Then $w(\tilde{M})$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ are now set functions defined in $\mathfrak{B}$, and w, w_+, w_- are measures, ω is a signed measure, and $w(\tilde{M}) = w_+(\tilde{M}) + w_-(\tilde{M}) = |\tilde{M}|$, $w_+(\tilde{M}) - w_-(\tilde{M}) = \omega(\tilde{M})$ for all $\tilde{M} \in \mathfrak{B}$. Finally, if J_+ is the set where $w'_+ = 1$, and $J_- = J - J_+$, then, for every set $\tilde{M} \in \mathfrak{B}$, we have $\omega(\tilde{M}J_+) = w(\tilde{M}J_+) = w_+(\tilde{M}J_+) \geqq 0$, $-\omega(\tilde{M}J_-) = w(\tilde{M}J_-) = w_-(\tilde{M}J_-) \geqq 0$, that is, J_+, J_- is a Hahn decomposition of J with respect to the signed measure ω [see, for these general concepts, P. R. Halmos, *Measure Theory*, 1950, p. 121].

Let $\Gamma = \{g\}$ denote the decomposition of $[a, b]$ into all maximal closed intervals g of constancy for $f(u)$ in $[a, b]$ (at most countably many, if any), and

all remaining single points g of $[a, b]$. The transformation $M_0 = s^{-1}(\tilde{M})$ transform the ring $\tilde{\mathfrak{B}}$ into the ring $\mathfrak{B}_0$ of all B-measurable sets $M_0 \subset [a, b]$ which are sums of elements $g \in \Gamma$. The same transformation transforms $w(\tilde{M})$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ into corresponding measures $v(M_0)$, $v_+(M_0)$, $v_-(M_0)$, $\mathfrak{V}(M_0)$ in the ring $\mathfrak{B}_0$, and the decomposition J_+, J_- of J into a Hahn decomposition H_+, H_- of $[a, b]$ with respect to the signed measure $\mathfrak{V}$. Finally, if $\mathfrak{B}$ is the ring of all B-measurable subsets $M \subset [a, b]$, and, for every $M \in \mathfrak{B}$, we denote by M_0 the minimum set $M_0 \supset M$, $M_0 \in \mathfrak{B}_0$, we may put $v(M) = v(M_0)$, and analogously for v_+, v_-, $\mathfrak{V}$. Then the set functions $v(M)$, $v_+(M)$, $v_-(M)$, $\mathfrak{V}(M)$ are defined for all sets $M \in \mathfrak{B}$, and v, v_+, v_- are measures, $\mathfrak{V}(M)$ is a signed measure in the ring $\mathfrak{B}$. Also, H_+, H_- is still a Hahn decomposition of $[a, b]$ for $\mathfrak{V}$, and we have $v(M) = v_+(M) + v_-(M)$, $\mathfrak{V}(M) = v_+(M) - v_-(M)$, $\mathfrak{V}(MH_+) = v(MH_+) = v_+(MH_+) \geqq 0$, $-\mathfrak{V}(MH_-) = v(MH_-) = v_-(MH_-) \geqq 0$ for all sets $M \in \mathfrak{B}$. (For modern developments on continuous varities in Euclidean spaces, see T. Rado, *Length and Area*, Amer. Math. Soc., 1948; L. Cesari, *Surface Area*, Princeton, 1956.)

MATHEMATICIANS IN THE MARKET PLACE

MINA REES, Hunter College

My theme is the extent and nature of the need for mathematicians in industry and government;* my concern is with some of the inadequacies in our college and university environment if we are to meet this need. In a world in which technology has assumed an importance which is drawing in its wake public realization of the need for sound scientific education and for the support of basic research, mathematicians of every level of originality and training are needed: industrial mathematicians to support advances in our technology and in our business operations, as well as research men to push forward the frontiers of mathematical research and lend imaginative insights to the corpus of mathematics, and teachers for our colleges and schools.

That adequate support and adequate recognition must be given to research in pure mathematics is accepted as obvious. That gifted young people must be given the fullest opportunity to taste its pleasures and be seduced by its charms is a first requirement of any program of mathematical education. But the times demand that additional groups of young mathematical scholars be trained who will seek their careers in the market place to provide business, industry, and government with the mathematical support for which they are finding increasing need. Fortunately the presence of increasingly large numbers of young people in our colleges makes the time propitious for us to take stock. Fortunately, too,

* I shall make little attempt to distinguish between government and industry. The existence of a number of industrial companies in which substantially all the business is directly or indirectly with the government makes such a distinction virtually impossible if we are discussing the nature of the work—though salary scales and the nature of tenure are usually quite distinct.

there are many mathematicians now engaged in industrial research who find it a fascinating and intellectually rewarding pursuit.

We need additional mathematicians not only in industry, but in the universities and in the secondary schools. Our present dilemma in staffing the schools and universities is caused partly by the drift of well-trained mathematicians to industry because of the combined lure of higher salaries and interesting work. The need is clear to train more able mathematicians and to raise academic salaries to a level competitive with those offered by industry. The second point I need not emphasize; but on the first point I should like to expand.

Since 1950 we have been producing an average of a little over 200 Ph.D.'s in mathematics a year in the United States (and the average before 1950 was considerably smaller). This is a staggeringly small number to fill the needs that exist. It is imperative that we attract more young people to careers in mathematics. But many able boys and girls for whom teaching is not attractive still have the idea that teaching is the only career to which training in mathematics leads; and the nature of industrial mathematics often is not understood by the teachers who provide the only contact young people have with mathematics. We need to acquaint our young people with the great variety of interesting careers open to mathematicians. Inevitably some who embark on the study of mathematics as a road to a career in industry will find themselves diverted by their enthusiasm for research in pure mathematics—a quid pro quo for those lost to business from the university. It may be worth noting that over one-third of living Americans with Ph.D.'s in mathematics came to mathematics from some other field—the most frequent fields being engineering, chemistry, and physics.

One additional point needs mention. The problem is not to steal all the ablest youth of the land from other pursuits but merely to maintain the position of mathematics. Mathematicians themselves have not recognized the strength of their position.

The occasion of these observations is not the latest Sputnik in the sky. On the contrary, in considering mathematical training, as in the larger educational universe, the family of Sputniks merely verifies what we knew already.

As a part of the recently completed *Survey of Research Potential and Training in the Mathematical Sciences*,* a subcommittee on nonteaching opportunities† gathered data and assessed attitudes that give some information on the following questions:

What is the size of the population employed as mathematicians in industry and government?

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* A Survey of Research Potential and Training in the Mathematical Sciences, University of Chicago, 1957.

† The members were: Hendrik Bode, Bell Telephone Laboratories, Inc.; A. H. Flax, Cornell Aeronautical Laboratory, Inc.; Alston Householder, Oak Ridge National Laboratory; Harlan Mills, General Electric Co.; Mina Rees, Hunter College, Chairman.

What is their quality as research men?

What is the nature of their mathematical activity in industry?

What training and personal qualifications are needed for success in industry?

Are there qualities or attitudes which distinguish industrial mathematicians as a group from university mathematicians?

I want now to report on some of these findings, and to put forth some recommendations based on the Subcommittee's report, as well as on interviews with persons engaged in mathematical research in industry and on discussions held by the Subcommittee.

The number of persons employed as mathematicians in nonacademic work is startling. Between 7000 and 8000 men and women are thought of by their employers as mathematicians, and nearly 7000 of these are employed without the Ph.D. Among Ph.D.'s over 900 are functioning as mathematicians—this in spite of the evidence of the Survey that there are at most 700, and probably between 600 and 700, American Ph.D.'s with original training in mathematics now engaged in nonteaching activities. The larger number reported by employers probably reflects several discrepancies of which the most important are (1) the fact that many Ph.D.'s originally trained as physicists or engineers are now operating as mathematicians, and (2) the fact that persons engaged in work making extensive use of statistics are thought of by their employers as mathematical statisticians, though their basic concern is with psychology, or economics, or some other field.

Also to be noted in this connection is the reply given by university departments of mathematics to the Survey question, "In what field or fields of mathematics do you most need personnel in order to fill gaps or to broaden your mathematical coverage?" About 100 vacancies were listed, with well over half lying in applied mathematical fields. Clearly the demand for Ph.D.'s with competence in applied mathematics or interest in the applications of mathematics is very great, and the need to re-examine our university environment is compelling.

With respect to non-Ph.D's employed as mathematicians we have a phenomenon with which all of us in academic work are familiar. The new A.B. or B.S. who goes out to a job paying a higher salary than his professor's is all too familiar; and the moral that professorial salaries must be raised is all too clear. It is, however, also clear that the colleges and universities are facing unprecedented demands from industry and government for graduates of mathematics curricula.

It may be worth a pause to note that the results of the Survey, indicating the size of the demand, are not old and familiar information, though many of us may be inclined to say we knew it all along. In the calendar year 1957, a thoroughly competent book by Blank and Stigler,* dealing with the demand and supply of scientific personnel and relying on the 1950 Census and the

* Demand and Supply of Scientific Personnel, David M. Blank and George J. Stigler, National Bureau of Economic Research, 1957.

latest available statistics (1951) concerning employment of mathematicians, reported that less than 13% of Ph.D. mathematicians were employed outside the colleges and universities. (The authors did suspect a bias in their data because of the emphasis, in the collection of data, on membership in professional societies.) The Survey data show nearly 23% of Ph.D. mathematicians in nonacademic positions, with the trend among recent Ph.D.'s approaching 30%.

What is the quality of these Ph.D.'s as research men? In spite of the prejudices on both sides (and these are very real) there seems to be very little evidence that there is much difference in quality between teachers and industrial mathematicians. The Survey Committee, like all other groups, had great difficulty in finding an acceptable measure of quality. The results do make clear that, since World War II, an increasingly large percentage of those with very high undergraduate grades are turning to nonacademic pursuits; and that the publication record of industrial mathematicians with postwar Ph.D.'s is somewhat better than the average for the whole group of Ph.D.'s. This is in spite of the fact that much original work done in industry and government does not find its way into publication because of the restrictions imposed by security or competition. A further note concerning quality is the election to the National Academy of Sciences, within the last two years, of two men who made their reputations in industrial mathematics.

With no evidence to indicate that the quality of the work done in industry is substantially different from that done at the university, it is none the less clear that the motivation, and often the nature of the research, is quite different in the two settings. Unfortunately there are few opportunities for academic mathematicians to understand the nature of the industrial mathematician's work and this lack of understanding is reflected in the impressions given to our students.

I was interested in receiving, a short time ago, the "Abbreviated Proceedings"* of a conference held at Oxford University, April 8–18, 1957, for school teachers and industrialists. This was supported by 18 groups, mostly industrial, a few government, and one American (The Office of Naval Research) to provide liaison between preparatory school teachers and industry. The school teachers were struck by the scale, complexity, and variety of mathematical work in industry. As a setting for some of the things I have to say, I want to quote from the introductory remarks by John Hammersley of Trinity College, Oxford, who organized the conference:

"Only a handful of school teachers have ever used mathematics in practice. Mathematical examination problems are usually considered unfair if insoluble or improperly described; whereas the mathematical problems of real life are almost invariably insoluble and badly stated, at least in the first instance. In real life, the mathematician's main task is to formulate problems by building an abstract mathematical model consisting of equations, which shall be simple

* Oxford Mathematical Conference (Abbreviated Proceedings), London, 1957.

enough to solve without being so crude that they fail to mirror reality. Solving equations is a minor technical matter compared with this fascinating and sophisticated craft of model-building, which calls for both clear, keen commonsense and the highest qualities of artistic and creative imagination."

A further quotation from the introductory remarks of W. E. Scott of the English Electric Company may also be enlightening:

"The growing complexity of some modern industries has . . . increased the opportunity for mathematicians to fit themselves in and become most valuable to industry. A main reason for this trend has been growing development costs. It is nowadays most expensive to carry out a new major project, and prohibitively so if the right approach is not made from the beginning. Thus the maximum amount of thought must be given to a design before it is built or to an experiment before it is started, and this often involves considerable theoretical study."

What is expected of the mathematician in industry? I speak here primarily of the Ph.D. mathematician, though some of my remarks will apply to the young college graduate.

The first consideration is the question of personality. Unless a man enjoys working with others, unless he is interested in considering other people's problems, unless he finds it interesting to evolve the appropriate mathematical model for handling situations that are often not correctly or clearly described, and to bring his mathematical maturity to bear on situations he has not himself selected he probably does not belong in industry. One relevant result of the Survey was not a surprise, but the statistics do serve to corroborate a strong prior impression. Group research is viewed more favorably by industrial mathematicians, and there is more of it among them than among university people.

An aspect of this collaborative feature of industrial research is the critical importance of the mathematician's ability to communicate with the nonmathematician. In industry the audience may be highly sophisticated scientists and engineers who do not, however, have the specific language and habits of thought of the mathematician. In government, the audience may well be a business man or a military officer. In both cases the effectiveness of the mathematician is largely conditioned by his ability to understand the other man's language, and to present to him, simply and comprehensibly, the power and the limitations of his craft. Thus the mathematician in a nonacademic environment requires to a marked degree some of the "teaching" interest and skill needed also by his academic colleagues. In order to understand the problems that are brought to him, he needs to acquire some of the basic ideas of the sciences and of engineering; in order to present his own ideas with cogency he needs experience in presenting mathematical ideas to nonmathematicians. To quote from the Survey Report: " . . . the applied mathematician ought to have significant subject matter knowledge in other scientific and technical areas. More important

than detailed subject matter knowledge as such, however, is a fairly extensive acquaintance with the "bare bones" logical structure of a number of areas. This is evidently a sort of background on the basis of which abstract mathematical model building and axiomatizing generally can be most readily and reliably applied in particular new situations. . . . The graduate student in mathematics who contemplates a nonacademic career would be well advised to acquire experience at the university in presenting his ideas to students and colleagues in other departments. At the minimum this experience should include engineering and science departments, but it is desirable that the social scientists also be included in the audiences."

The report of the Oxford Conference includes this statement by D. G. Owen of the British Iron and Steel Federation:

"We also need, especially in our mathematicians, a practical outlook and good personality: a problem may take three months to formulate and solve, but unless the results are 'sold' to management and made to work *in practice* the job cannot be regarded as complete. High quality investigators . . . are wanted, and they must be able and willing to talk with laymen in acceptable language."

What of the specific mathematical training needed for work in industry? The striking thing about modern applications of mathematics is that fields that were thought of as highly abstract even a decade ago have proved to be highly applicable in the modern setting. There is the obvious relevance of many parts of analysis, including differential equations, particularly of the nonlinear variety, to the new problems of propulsion and aerodynamics of high speed aircraft on the one hand, and, on the other, to problems arising from nuclear weapons research and other explosion phenomena, from radar and other electronics problems. If I may quote still another time from the Oxford Conference report, P. L. Taylor of Metropolitan-Vickers Electrical Co. Ltd., in his paper on "The Mathematics of Linear Electric Circuit Theory" has this to say:

"Since the properties of a circuit are essentially independent of the geometrical arrangement of the connecting wires—they may be arranged at will provided only that no connections are broken or new ones made—it is evident that it is the topological properties of the linear graph that are of interest."

And further:

"In manipulating the system of equations that describe a circuit the ideal tool is matrix algebra."

The application of Boolean algebra to computer design, of group theory to communications theory, of number theory to numerical analysis, and of noncommutative algebra and group theory to nuclear physics leaves one with the quite correct impression that almost any mathematics is applicable, and that broadly educated mathematicians are needed.

I have chosen to quote from a British report to emphasize the similarity of the situation in England and the United States. In our country, too, a sampling

of problems confronted in industry shows a broad spectrum of mathematical disciplines in use in the attempt to find solutions. On a recent visit to the Bell Telephone Laboratories, I was told of a few of the problems in which some members of the mathematical research group are interested. One of these originated in the Financial Department of the American Telephone and Telegraph Company, and concerned the charges for private wire services. The mathematical result that was derived at Bell Laboratories gave a new theorem in the theory of trees. Interestingly enough the same result seems to have been found at about the same time at the Mathematisch Centrum in Amsterdam when, in the design of a new computer, the old computer was asked to determine what wiring would keep the total length of wire as small as possible. In the Telephone Company's problem, the Federal Communications Commission requires that the charges be based on the shortest network that can be constructed connecting plants at several locations; in the computer problem, the effort is to minimize the network of wires connecting a collection of pins. The theorem previously existing in the literature required that the minimum distance between *any* two points be found—a requirement poorly adapted to computation. The new result gives a systematic procedure that can easily be translated into a machine program. We have here a situation that illustrates the new requirement often found in the industrial applications of mathematics—to find a new theorem which will enable us to exploit the speed of electronic computers. For industrial applications, the computability of a method is even more important than elegance is for a proof in pure mathematics.

Another Bell Laboratories problem concerns communication in the presence of noise, and involves quantizing the sending of bits of information. The problem is to translate into usable form an existence proof and asymptotic theorems due to Shannon. Though the problem is unsolved, some progress has been made using the theory of groups.

A new kind of demand for mathematicians has been produced by the current expansion of the use of operations research in industry. This is a type of nonacademic employment that relies less on specific techniques than do the traditional fields of applied mathematics. A large expansion in this type of activity is in process, with a significant increase in the opportunities for mathematicians. When I was asked last year by the publisher of *Career* magazine to suggest mathematicians who might write accounts of mathematics in industry for high school students, I gave him the names of a group of men, who, I thought, would provide a broad spectrum of experience, and not emphasize computing exclusively. What was my dismay to find when the magazine appeared that there was substantially nothing in it but operations research. Even the communications industry and the aircraft industry—old standbys for classical applied mathematics—are now emphasizing operations research.

I have not mentioned the familiar fields of statistics and computing. The uses of statistics are expanding in academic research as well as in industrial work, and the need for statisticians who can carry on independent work con-

tinues to grow. In England, the emphasis on more training in statistics in the schools parallels some of the suggestions of our own Commission on Mathematics. One company, Imperial Industries, Ltd., reports that they hope to meet their shortage of statisticians by "recruiting mathematicians straight from graduation at universities and then sending them back to the most appropriate university departments for postgraduate training in mathematical statistics"* after they have spent a little time at work getting an idea of the sort of problems faced by the industry.

In computing, the availability of large fast machines has changed the face of industrial research, and we shall be hard pressed to provide trained users of these machines. As the telephone company's leased lines problem suggests, one of the boundary conditions imposed on many industrial problems is the ease with which the proposed formulation or solution yields to evaluation by automatic computation. Moreover the availability of large-scale fast computers has multiplied the number of problems for which industry seeks mathematical solution, and has given industry a powerful tool for attacking some problems for which no analytical procedures were formerly sought. One of the problems currently being reduced to numerical solution at Westinghouse may serve to illustrate the point.†

At an appliance plant in Ohio, a problem was posed very simply: "Given a product that is to be manufactured on an assembly line, what is the best way to arrange the assembly line in order to minimize the amount of human labor." There are boundary conditions, like the existence of zones where certain operations must be performed, and the permutability of some types of operation and nonpermutability of others. Traditionally, such assembly lines have been set up by industrial engineers, with no certainty that the solution which has been designed is optimal. The mathematician who was consulted on this problem used point set theory to find an optimal solution (not unique); and proceeded to show, by hand computation, that the mathematical solution for an existing assembly line would reduce the number of people needed for the job from 25 to 22 —a result of substantial interest for management. The programming of the solution for a digital computer makes the solution available to the company's various management divisions which can get the answers to their own specific problems with digital computer speed.

I have referred several times to the situation in England. Though this is in many ways similar to our own, it is not the same. The British, as well as the Russians, have been conspicuous in the high quality of their research in applied mathematics. It probably is not an accident that this excellence is found in a country where the traditional training of a mathematician includes substantial formal work in applied fields. The delightful essay of Littlewood on "A Mathe-

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* *Loc. cit.*, p. 80.

† Based on a description provided by M. Ostrofsky, Manager, Department of Mathematics, Westinghouse Electric Corp.

matical Education"* discloses without comment, and in a matter-of-fact way, that his education at Cambridge, beginning in 1903, included the study of dynamics, hydrodynamics, spherical harmonics, electricity, and the foundations of mechanics. This tradition continues. The same subjects will be found in most university schedules today. For example, at one of the colleges at the University of London, there are four hours of lecture in applied mathematics during a student's first year at the University, and six hours during his second year. During his third year he may elect to specialize in pure mathematics, or in applied mathematics, or to continue in both.

Whether American universities will wish to encourage formal course work in applied mathematics is problematical, and whether they should is controversial. But students should be given the opportunity to learn something about the nature of industrial mathematics. The training needed for effective work as an industrial mathematician has been described to me by Hendrik Bode, on the basis of his experience at Bell Telephone Laboratories.† In his judgment, an applied mathematician needs to serve one, or perhaps several, interneships. For example, in these days of automatically sequenced computing machines a thorough acquaintance with numerical methods, standard approximating procedures, and the various ways of attacking problems numerically are obviously of special importance. This is the sort of ability which is best learned by actual practice in a computing center. The computing facility need not be a large one—in fact there may be a positive advantage in having to use one's ingenuity to overcome mechanical inadequacies; but a first-hand acquaintance with numbers, of the sort that is developed only by a certain amount of laboratory work, is almost essential.

Another significant interneship could be spent in a statistical center. This would give the aspiring applied mathematician a better appreciation than he could probably have otherwise of the reliability of the factual statements to which he will be subjected later, and would also give him a feeling for the probable significance, or lack of significance, of a given element of a logical framework in a particular situation.

Another profitable, if brief, interneship might be served as a participant in an actual functioning applied mathematics group in an industrial or similar environment. This would be particularly useful if it occurred before the end of the formal training period. It might, of course, be possible to find ways of meeting most, if not all, of these interneship requirements in a single place.

The young applied mathematician should also find time, if possible while he is still at the university, to participate to some extent in seminars or research activities in other disciplines.

Although there are few university departments of mathematics which are ideal for this type of training, a sympathetic point of view on the part of the department as a whole goes far toward making adequate training possible. In a

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* J. E. Littlewood, A Mathematician's Miscellany, London, 1953.

† The following five paragraphs are paraphrased from a letter from Hendrik Bode.

large university especially, where there is always ample and diversified research activity, the young applied mathematician should be able to find most of what he needs if he is given reasonable support and encouragement.

The basic need is for a cordial atmosphere at the university. For decades chemistry departments have accepted as part of their established function the training of industrial chemists. More recently, since World War II, the physicists have come to accept a similar role. At present three-fourths of our chemistry Ph.D.'s are seeking their careers in industry, and three-fifths of our physicists. Mathematics is now emerging into a situation like that of chemistry and physics. While it is true that there are university professors who believe that good mathematics is useless mathematics, it is interesting that the statistics of the Survey show that Harvard, Princeton and Yale perform in an average manner in the number of Ph.D.'s they send into nonacademic mathematical pursuits. But it is also interesting that, although the number of Ph.D.'s going into industrial work is increasing, few of them were aware of the opportunities in these fields when they started graduate work. Better information is needed in the schools, the colleges and the universities about the nature of the work mathematicians do in industry.

On this front some progress is being made. On the secondary level, the Committee on Applications of Mathematics of the National Research Council, in collaboration with the National Council of Teachers of Mathematics, is preparing for publication a booklet of profiles of successful young mathematicians, most of them in industrial positions. This should be completed within a year; it will be very widely distributed in the secondary schools of the country. At the college and university level, also, some progress is indicated by the replies to one of the Survey questions. Of the university and college professors who responded, a substantial number indicated that consultation with industry or government was among their four most important contributions to mathematics. Thus there is some flow of information about industrial mathematics into university faculties. But more needs to be done. The flow of information from university to industry is accomplished in significant part by the summer visits of many professors to industrial laboratories. These visits often have an instructional flavor, with university men bringing up-to-date or sophisticated methods to some of the young mathematicians who have been recruited into industry without extensive training. Often, however, the professor is participating with senior industrial colleagues in an enriching experience. More extended work experience by academic people, and extended loans to universities by industry would benefit both. More general use of visits by students to industrial establishments and by industrial mathematicians to college and university classes should be explored. There are already extensive fellowship programs in some industries which provide selected employees with an opportunity to do advanced graduate work at neighboring universities at company expense; and there are companies which provide lecturers in applied topics to near-by universities. There are also extensive in-service graduate training programs in some companies.

The Survey Subcommittee recommended that other ways be sought to establish better cooperation between university and industry. The possibility of summer institutes in areas of applied interest should be explored, with participation by academic and nonacademic mathematicians. This is a problem which I hope will be one of the first items on the agenda of the newly established Conference Organization of the Mathematical Sciences. Among activities that the Society for Industrial and Applied Mathematics (SIAM) might well undertake, I have two suggestions. The first draws its inspiration from the economists. The Foundation for Economic Education, now in its eleventh year, provides summer fellowships, intended merely to pay expenses, which enable professors to visit one of the participating industries for a period of about six weeks, and become acquainted with management aspects of the company. Last year there were 80 colleges and universities that participated, and 56 business firms. There is now at least one aircraft company which makes similar arrangements for visits by university mathematicians. An organized effort like this might well prove mutually beneficial. Similar activities for secondary school mathematics teachers have been carried on in the Los Angeles area.

Probably easier to achieve would be a visiting lecturer program similar to that administered by the Mathematical Association. If SIAM would provide for distinguished lecturers from industry to visit the colleges and universities, on a schedule comparable but suitably different from that used by the Association, it is probable that an important new element might be introduced into the atmosphere on many campuses.

One of the few significant differences between nonacademic and academic mathematicians which the Survey disclosed, was the dissatisfaction of the industrial mathematicians with the publication policies of the mathematical world. This is particularly significant since between 25 and 30% of the members of the American Mathematical Society are employed outside the university. Various suggestions have been made to try to remedy this situation; and it is pleasant to note that the Society has recently added the Journal of SIAM to the group of its subsidized publications, in the hope that this Journal may provide another desirable publication outlet for some of the research results of nonacademic mathematicians.

It is important that all members of the mathematical community exert their maximum effort to eliminate the friction that exists between pure and applied mathematicians. As F. J. Weyl says in his excellent report on Applied Mathematics in the United States:*

"Training and research in mathematics has now become of the same immediate importance to the public welfare as training and research in the experimental

* Survey of Training and Research in Applied Mathematics in the United States conducted by the National Research Council, F. J. Weyl, Principal Investigator, November, 1954. Published by the Society for Industrial and Applied Mathematics, Philadelphia, 1956.

sciences. Rather than indirectly, as in the past, through services rendered to the other sciences along the way of their development, it is making progressively more of its contributions to the concerns of the day directly and in concert with the other sciences .... This not only gives a broadened scientific scope and golden new opportunities to mathematics, but it also throws an added burden of social responsibility on its practitioners, making it of importance that the applied aspects of mathematics and those who have concern therewith are recognized and heeded in the representative organizations of American mathematics."

Mathematics is an entity. The welfare of each part is essential to the other. Salaries at the university advance with the increased demand for mathematicians in industry; and financial support for research in pure mathematics accompanies support for applied research. Wisdom and far-sightedness demand that all mathematicians join forces in attracting more students to the study of mathematics, and in offering them a broad view of the role of mathematics in our civilization—in research, in teaching, and in industrial applications. We should find a way to put an end to the attitude among university mathematicians that industrial mathematicians are second-class citizens; and we should recognize them for what they are: a source of power for mathematics, a reason for the respect in which pure mathematics is held by many who do not understand it, and sometimes a leavening influence in determining the directions of interesting new mathematical research.

COMPLETENESS AND PARSEVAL'S EQUATION

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For a fixed finite closed interval $[a, b]$, let L^2 denote the space of all real-valued functions that are defined and measurable on $[a, b]$ and whose squares are Lebesgue integrable there, with identification of functions equal almost everywhere [1]. If f and g are any two members of L^2, then the *inner product* (f, g), the *norm* $\|f\|$, and the *distance* $d(f, g)$ are defined by the equations $(f, g) = \int_a^b f(x)g(x)dx$, $\|f\| = (f, f)^{1/2}$, $d(f, g) = \|f-g\|$. A sequence $\{\phi_n\}$ of members of L^2 is *orthonormal* if and only if $(\phi_m, \phi_n) = 0$ or 1 according as $m \neq n$ or $m = n$. The *Fourier series* of a member f of L^2 with respect to an orthonormal sequence $\{\phi_n\}$ is the series $\sum \alpha_n \phi_n$, where the numbers α_n are the *Fourier coefficients* $\alpha_n = (f, \phi_n)$, $n = 1, 2, \cdots$.

Certain facts hold for a Fourier series without further assumption [2]. One is Bessel's inequality

$$\sum \alpha_n^2 \leq \|f\|^2. \tag{1}$$

Another is the "least squares" property of the Fourier coefficients, which states

that the distance between a given f and any linear combination of a fixed finite set of $\{\phi_n\}$ is minimized if the coefficients are the Fourier coefficients.

For questions of convergence, point-wise or in the mean, and Cesàro summability [3, 4] it is important to include a further hypothesis guaranteeing that in some sense or other there are "enough" members in the orthonormal family. This added hypothesis can be expressed in many equivalent forms including, in particular, the following six:

1. For any $f \in L^2$ the Fourier series $\sum \alpha_n f_n$ *converges in the mean* to f:

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} \alpha_n f_n - f \right\| = 0.$$

2. The finite linear combinations of members of $\{\phi_n\}$ are *dense* in L^2.
3. *Parseval's equation* holds for any $f \in L^2$:

$$\sum \alpha_n^2 = \|f\|^2.$$

4. The sequence $\{\phi_n\}$ is *complete* or *maximal*; that is, there is no strictly larger orthonormal sequence containing $\{\phi_n\}$.
5. The only member of L^2 orthogonal to every member of $\{\phi_n\}$ is 0.
6. A member of L^2 is uniquely determined by its Fourier coefficients.

The proof that all of these six conditions are equivalent rests in part on the completeness of L^2 as a metric space [2]. The question now is the exact extent to which the completeness of L^2 is needed. To make the question more precise, let us replace the space L^2 by the (incomplete) subspace R^2, the Lebesgue integral being replaced by the standard proper Riemann integral. Which of the preceding six conditions imply which others? At least a partial answer is based on standard and rather elementary techniques [2]. This is that the first three are equivalent, the last three are equivalent, and the first three imply the last three:

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6. \tag{2}$$

The one remaining unanswered question is whether the central implication in (2) can be reversed. It will be the purpose of the remaining portion of this paper to show that *the reverse implication does not hold*:

$$3 \not\Leftarrow 4, \tag{3}$$

and that a simple demonstration can be provided by extending the space R^2 to its completion L^2.

For convenience of notation we shall denote members of the two spaces R^2 and L^2 by lower-case and upper-case letters, respectively: $f, g, h \in R^2$, $F, G, H \in L^2$. Let F be an arbitrary member of L^2 not equivalent to a member of R^2 (for example, F might be an unbounded function, or the characteristic function of a Cantor set of positive measure), normalize F with an appropriate factor so

that $\|F\|=1$, and let Π be the orthogonal complement of F in L^2 [2].

LEMMA. *The set $R^2\cap\Pi$ is dense in Π.*

Proof. Let G be an arbitrary member of Π, and let $\{f_n\}$ and $\{h_n\}$ be sequences in R^2 such that $f_n\to F$, $h_n\to F+G$, and assume that $\|f_n-F\|<1$ for all n. Define $g_n=h_n-\lambda_n f_n$, where λ_n is determined so that $g_n\in\Pi$: $\lambda_n=(h_n, F)/(f_n, F)$. As $n\to\infty$, by the continuity of the inner product, $(f_n, F)\to(F, F)=1$, $(h_n, F)\to(F+G, F)=(F, F)=1$, and $\lambda_n\to 1$. Therefore $g_n\to(F+G)-1(F)=G$.

Since the space L^2 is separable [1], so is Π, and hence there exists a sequence $\{f_n\}$ in $R^2\cap\Pi$ dense in Π. Assuming without loss of generality that no f_n is a linear combination of members of $\{f_n\}$ with subscripts less than n, we can apply the Gram-Schmidt process [2] and obtain an orthonormal sequence $\{\phi_n\}$ of members of $R^2\cap\Pi$. Since this sequence lies in the closed linear subspace Π, which is not dense in L^2, it follows that the finite linear combinations of members of $\{\phi_n\}$ cannot be dense in L^2. Therefore these finite linear combinations are not dense in the subspace R^2 (which *is* dense in L^2), and the sequence $\{\phi_n\}$ does not satisfy any of the first three of the six listed conditions, applied to the space R^2. On the other hand, if a member H of L^2 is orthogonal to every ϕ_n it is orthogonal to every member of Π and hence must be a multiple of F. Therefore if H belongs to R^2 it must equal 0, and the sequence $\{\phi_n\}$ is complete in R^2 and satisfies the last three of the six listed conditions. This completes the proof.

The preceding argument applies to any incomplete linear subspace of L^2, and we conclude that *Parseval's equation is a consequence of completeness for orthonormal sequences only in the presence of completeness for the linear space concerned.*

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MATHEMATICAL NOTES

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A WRONSKIAN

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Let u and v be differentiable functions of x and consider the Wronskian of the set of functions u, uv, uv^2, $\cdots$, uv^n. We denote the Wronskian by $W(u, uv, uv^2, \cdots, uv^n)=|a_{sj}|$, $s, j=1, 2, \cdots, n+1$, where $a_{sj}=D^{s-1}(uv^{j-1})$

and $D \equiv d/dx$. We will show that

$$W(u, uv, uv^2, \cdots, uv^n) = u^{n+1} \cdot [Dv]^{(n(n+1))/2} \cdot \prod_{i=1}^{n} i! \tag{1}$$

To establish (1), we will show that the Wronskian can be transformed into a triangular form, *i.e.*, every element above the principal diagonal is zero.

Proceeding from right to left, we add to the elements of the $(j+1)$th column $(-v)$ times the elements in the jth column, starting with $j=n$ and continuing on, respectively, with $j=n-1, n-2, \cdots, 1$. The above steps will be called a *process*. We repeat the *process* again, where now $j=n, n-1, \cdots, 2$, respectively. All told, we repeat the *process* n times, where in the pth process, $p \leqq n$, $j=n, n-1, \cdots, p$. We note that the elements of the jth column assume their final values at the completion of the $(j-1)$th *process*.

Let $a_{sj}^{(p)}$ denote the element in the sth row and jth column after p applications of the *process*, where $j \leqq n+1$, $p \leqq j-1$. Then, we have

$$\begin{aligned}
a_{sj}^{(1)} &= a_{sj} - v a_{sj-1} \text{ (note that } a_{1j}^{(1)} = 0 \text{ for } j > 1) \\
a_{sj}^{(2)} &= a_{sj}^{(1)} - v a_{sj-1}^{(1)} = a_{sj} - 2v a_{sj-1} + v^2 a_{sj-2} \\
a_{sj}^{(3)} &= a_{sj}^{(2)} - v a_{sj-1}^{(2)} = a_{sj} - 3v a_{sj-1} + 3v^2 a_{sj-2} - v^3 a_{sj-3}.
\end{aligned}$$

Denoting, as usual, $\binom{p}{k} = p!/k!(p-k)!$, we claim that $a_{sj}^{(p)} = a_{sj}^{(p-1)} - v a_{sj-1}^{(p-1)}$ satisfies the relation

$$a_{sj}^{(p)} = \sum_{k=0}^{p} (-1)^k \binom{p}{k} v^k a_{sj-k}. \tag{2}$$

We may prove (2) by mathematical induction on p. (2) is true for $p=1$. Suppose now that (2) is true for $p=P$. Then

$$a_{sj}^{(P+1)} = a_{sj}^{(P)} - v a_{sj-1}^{(P)} = \sum_{k=0}^{P} (-1)^k \binom{P}{k} v^k a_{sj-k} - \sum_{k=0}^{P} (-1)^k \binom{P}{k} v^{k+1} a_{sj-(k+1)}.$$

In the second sum, let $t=k+1$. Since $\binom{P}{k} = 0$ for $k > P$, $k < 0$, we have

$$\begin{aligned}
a_{sj}^{(P+1)} &= \sum_{k=0}^{P+1} (-1)^k \binom{P}{k} v^k a_{sj-k} + \sum_{t=0}^{P+1} (-1)^t \binom{P}{t-1} v^t a_{sj-t} \\
&= \sum_{k=0}^{P+1} (-1)^k \left[\binom{P}{k-1} + \binom{P}{k} \right] v^k a_{sj-k} = \sum_{k=0}^{P+1} (-1)^k \binom{P+1}{k} v^k a_{sj-k}.
\end{aligned}$$

This completes the proof of (2). Let $b_{sj} = a_{sj}^{(j-1)}$; then from (2), for $p=j-1$, we have

$$b_{sj} = \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} v^k D^{s-1}(uv^{j-1-k}), \quad s, j = 1, \cdots, n+1. \tag{3}$$

We claim now that

$$b_{sj} = 0 \text{ if } j > s, \tag{4}$$

$$b_{ss} = (s-1)!\cdot u(Dv)^{s-1}, \qquad s = 1, 2, 3, \cdots, n+1. \tag{5}$$

From Leibniz's rule, we have

$$D^{s-1}(uv^{j-1-k}) = \sum_{m=0}^{s-1} \binom{s-1}{m} D^{s-1-m}(uv^{j-1}) D^m(v^{-k});$$

and so

$$b_{sj} = \sum_{m=0}^{s-1} \binom{s-1}{m} D^{s-1-m}(uv^{j-1}) \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} v^k D^m(v^{-k}). \tag{6}$$

Now, $v^k D^m(v^{-k})$ is a polynomial in k of degree m, *i.e.*,

$$v^k D^m(v^{-k}) = (-v)^{-m}(Dv)^m k^m + \sum_{i=0}^{m-2} g_i(v) k^{m-1-i}, \tag{7}$$

where the explicit expressions for $g_i(v)$ are of no concern. Recalling the well-known identity,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m = \begin{bmatrix} 0, & \text{if } m < n \\ n!(-1)^n, & \text{if } m = n \end{bmatrix}, \tag{8}$$

we find from (6), using (7) and (8), that $b_{sj}=0$ if $j>s$; for $j=s$, there is only one nonzero term, when $m=s-1$. Thus, $b_{ss}=(-v)^{-(s-1)}\cdot(Dv)^{s-1}\cdot(-1)^{s-1}\cdot[(s-1)!]\cdot uv^{s-1}=(s-1)!\cdot u(Dv)^{s-1}$. Since the determinant $|b_{sj}|$ is triangular, we have

$$W(u, uv, uv^2, \cdots, uv^n) = \prod_{s=1}^{n+1} b_{ss},$$

thus yielding (1) as claimed.

For $v=u$, $n=k-1$, we obtain from (1)

$$W(u, u^2, u^3, \cdots, u^k) = u^k\cdot[Du]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!. \tag{9}$$

From (1), with $v=1$, $W(u, u, \cdots, u)\equiv 0$. Some applications of (1) and (9) are now given.

(i) For linear differential equations with constant coefficients, the following Wronskians may be of interest:

If $u=e^{ax}$, $v=x$, then

$$W(e^{ax}, xe^{ax}, x^2e^{ax}, \cdots, x^ne^{ax}) = e^{a(n+1)x}\cdot \prod_{t=1}^{n} t!. \tag{10}$$

If $a=0$ in (10), then

$$W(1, x, x^2, \cdots, x^n) = \prod_{t=1}^{n} t!. \tag{11}$$

If $u=e^{ax}\cos bx$, $v=x$, then

$$W(e^{ax}\cos bx, xe^{ax}\cos bx, \cdots, x^n e^{ax}\cos bx) = (e^{ax}\cos bx)^{n+1}\cdot \prod_{t=1}^{n} t!. \tag{12}$$

Since the above Wronskians are defined for $x=0$, they satisfy Abel's identity for $x=0$.

(ii) Using (9), we find:

If $P(x)$ is any polynomial in x, then

$$W(P(x), P^2(x), \cdots, P^k(x)) = P^k(x)\cdot [DP(x)]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!. \tag{13}$$

If $f(x)$ is a differentiable function of x, then

$$\begin{aligned} &W\left(\int_a^x f(t)dt, \left[\int_a^x f(t)dt\right]^2, \cdots, \left[\int_a^x f(t)dt\right]^k\right) \\ &\quad = \left[\int_a^x f(t)dt\right]^k \cdot [f(x)]^{(k(k-1))/2}\cdot \prod_{m=1}^{k-1} m!. \end{aligned} \tag{14}$$

Since $\int_a^x f(t)dt = (x-a)f(u)$, where $a<u<x$ or $x<u<a$, we have

$$\lim_{x\to a}\left[\frac{W\left(\int_a^x f(t)dt, \cdots, \left[\int_a^x f(t)dt\right]^k\right)}{(x-a)^k}\right] = [f(a)]^{(k(k+1))/2}\cdot \prod_{m=1}^{k-1} m!. \tag{15}$$

If $u=f(x)e^{mx}$, where m is a constant, $m\neq 0$, then

$$\begin{aligned} &W(f(x)e^{mx}, f^2(x)e^{2mx}, \cdots, f^k(x)e^{kmx}) \\ &\quad = [f(x)]^k\cdot e^{(k(k+1)mx)/2}\cdot [mf(x) + Df(x)]^{k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!. \end{aligned} \tag{16}$$

Factoring out e^{jmx} from the jth column of the Wronskian, we have $e^{\sum_{j=1}^{k} jmx} = e^{k(k+1)mx/2}$. Thus, we find the value of the resultant determinant of order k:

$$\begin{aligned} |c_{sj}| &= \left|\sum_{i=0}^{s-1} (jm)^i \binom{s-1}{i} D^{s-1-i}[f(x)]^j\right| \\ &= [f(x)]^k\cdot [mf(x) + Df(x)]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!, \end{aligned} \tag{17}$$

where $s, j=1, 2, 3, \cdots, k$.

(iii) From (1), if $n=k$, $u=x^n$, $v=\ln x$, then

$$W(x^n, x^n \ln x, \cdots, x^n \ln^k x) = x^{((2n-k)(k+1))/2} \cdot \prod_{t=1}^{k} t! \tag{18}$$

(18) solves the elementary problem E1213 [1956, 253] of this MONTHLY, vol. 63.

(iv) Let $f=f(x)$ and $g=g(x)$ be differentiable functions of x. Then we find that

$$\begin{aligned} W(f^n, gf^{n-1}, g^2f^{n-2}, \cdots, g^{n-1}f, g^n) &= (f^n)^{n+1}\left[D\left(\frac{g}{f}\right)\right]^{(n(n+1))/2} \prod_{t=1}^{n} t! \\ &= (g^n)^{n+1}\left[-D\left(\frac{f}{g}\right)\right]^{(n(n+1))/2} \prod_{t=1}^{n} t!. \end{aligned} \tag{19}$$

If we define $u=f^n$, $v=g/f$, then the first form of (19) is obtained from (1). Noting that $D(g/f) = -(g/f)^2 D(f/g)$, we obtain the second form of (19). We observe that the first row of the Wronskian in (19) is obtained as the successive terms of the expansion $(f+g)^n$, with omission of the binomial coefficients.

SUMMING OF TRIGONOMETRIC SERIES WITH COEFFICIENTS WHICH HAVE A PERIODIC FACTOR

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Introduction. There are certain Fourier series which converge to elementary functions. The question arises as to whether these series still converge to elementary functions if certain terms are removed. For example, the following series

$$F(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

converges to

$$\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad \text{when} \quad 0 \leqq x \leqq 2\pi. \tag{1}$$

Can we find an elementary expression for the series

$$\frac{\cos x}{1^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \frac{\cos 9x}{9^2} + \frac{\cos 11x}{11^2} + \cdots ? \tag{2}$$

To answer this, we first define a function $f(x)$ to be in a class S if it can be expressed in open intervals as a finite sum of terms of the type Ax^le^{bx} where A and b are complex constants and l is a nonnegative integer.

The following theorem will answer the above problem:

THEOREM. *If $\sum_{n=1}^{\infty} r(n)$ sin nx is in S and $f(n)$ is an even function of period p, then $\sum_{n=1}^{\infty} r(n)f(n)$ sin nx is in S. If $\sum_{n=1}^{\infty} r(n)$ cos nx is in S and $f(n)$ is an odd function of period p, then $\sum_{n=1}^{\infty} r(n)$ cos nx is in S.*

Proof. If $f(n)$ is an even function of period p, then it can be expanded as (see [1])

$$(3)\qquad f(n) = \sum_{k=1}^{p} a_k \cos\frac{2n\pi k}{p}, \quad \text{where}$$
$$a_k = \frac{1}{p}\sum_{m=1}^{p} f(m)\cos\frac{2m\pi k}{p}.$$

The series $\sum_{n=1}^{\infty} r(n)f(n)\sin nx$ can be written as

$$\sum_{k=1}^{p} a_k \sum_{n=1}^{\infty} r(n)\cos\frac{2n\pi k}{p}\sin nx$$
$$= \frac{1}{2}\sum_{k=1}^{p} a_k \sum_{n=1}^{\infty} r(n)\left[\sin n\left(x+\frac{2\pi k}{p}\right)+\sin n\left(x-\frac{2\pi k}{p}\right)\right].$$

Since the series $F(x)=\sum r(n)\sin nx$ is in S, the above series reduces to

$$\frac{1}{2}\sum_{k=1}^{p} a_k\left[F\left(x+\frac{2\pi k}{p}\right)+F\left(x-\frac{2\pi k}{p}\right)\right],$$

which, from the definition, is still in S. A similar argument holds if $f(n)$ is an odd function. The theorem is valid, in particular, if, in the sine series, $r(n)$ is an odd function or in the cosine series $r(n)$ is an even function. (See [2] or [3]).

Example. Consider the series

$$(4)\qquad G(x) = \frac{\cos x}{1^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\frac{\cos 9x}{9^2}+\cdots+\frac{f(n)\cos nx}{n^2}+\cdots,$$

where
$$f(n) = \begin{cases} 1 & \text{if } n \equiv 1 \text{ or } 4 \bmod 5, \\ 0 & \text{if } n \equiv 0, 2, \text{ or } 3 \bmod 5. \end{cases}$$

Since $f(n)$ is an even function of period 5, it has the expansion

$$f(n) = \frac{2}{5}+\frac{4}{5}\left[\cos\frac{2\pi}{5}\cos\frac{2\pi n}{5}+\cos\frac{4\pi}{5}\cos\frac{4\pi n}{5}\right].$$

Using the above theorem, this sums to

$$G(x) = \frac{2}{5}F(x)+\frac{2}{5}\cos\frac{2\pi}{5}\left[F\left(x+\frac{2\pi}{5}\right)+F\left(x-\frac{2\pi}{5}\right)\right]$$
$$+\frac{2}{5}\cos\frac{4\pi}{5}\left[F\left(x+\frac{4\pi}{5}\right)+F\left(x-\frac{4\pi}{5}\right)\right],$$

where

$$F(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$$

if $0 \leqq x \leqq 2\pi$, and $F(x+2\pi) = F(x)$ if $-\infty < x < \infty$.

Sometimes the original series is not in S, whereas the series obtained by multiplying by a periodic factor is in S. For example,

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is not in S, whereas

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \cdot f(n),$$

where

$$f(n) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 3, \\ 1 & \text{if } n \equiv 1 \bmod 3, \\ -1 & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

has the sum

$$\frac{2}{3} \sin \frac{2\pi}{3} \left[F\left(x - \frac{2\pi}{3}\right) - F\left(x + \frac{2\pi}{3}\right) \right]$$

which is in S.

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A NOTE ON LINEAR DIFFERENCE EQUATIONS

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This note contains a modification of the method given by Friedman* for solving homogeneous equations which yields a formal solution of the non-homogeneous linear difference equation. Strangely enough, I have not been able to find this solution in standard reference books. For simplicity, the present discussion will be confined to the second-order equation, but the modification for higher order equations is obvious.

Assume that $f(0)$ and $f(1)$ are given as initial conditions, and consider the

* B. Friedman, Principles and Techniques of Applied Mathematics, New York, 1956, pp. 122–123. Milne-Thompson gives essentially the same method.

linear difference equation

$$f(x+2) = p(x)f(x+1) + q(x)f(x) + r(x), \tag{1}$$

and the vectors $(f(x+2), f(x+1), 1)$ and $(f(x+1), f(x), 1)$. By using equation (1) we find that

$$\begin{aligned}(f(x+2), f(x+1), 1) &= (p(x)f(x+1) + q(x)f(x) + r(x), f(x+1), 1)\\ &= (f(x+1), f(x), 1)\begin{pmatrix} p(x) & 1 & 0\\ q(x) & 0 & 0\\ r(x) & 0 & 1\end{pmatrix}.\end{aligned}$$

This constitutes a homogeneous recurrence relation, and

$$(f(n), f(n-1), 1) = (f(1), f(0), 1)\left[\prod_{k=0}^{n-2}\begin{pmatrix} p(k) & 1 & 0\\ q(k) & 0 & 0\\ r(k) & 0 & 1\end{pmatrix}\right] \tag{2}$$

with the matrices in the product arranged so that the factor for $k=j$ is immediately to the right of the factor for $k=j-1$. The first components of these vectors can be isolated by forming the scalar product with the column vector (1, 0, 0) to get

$$f(n) = (f(1), f(0), 1)\left[\prod_{k=0}^{n-2}\begin{pmatrix} p(k) & 1 & 0\\ q(k) & 0 & 0\\ r(k) & 0 & 1\end{pmatrix}\right]\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix} \tag{3}$$

as the solution of (1).

Note that each matrix in the product can be written in terms of blocks by distinguishing the last row and column in the form

$$\begin{pmatrix} p(k) & 1 & 0\\ q(k) & 0 & 0\\ r(k) & 0 & 1\end{pmatrix} = \begin{pmatrix} P_k & 0\\ R_k & 1\end{pmatrix} \quad \text{with} \quad P_k = \begin{pmatrix} p(k) & 1\\ q(k) & 0\end{pmatrix}, \quad R_k = (r(k) \quad 0).$$

Hence the matrix product will have the form

$$\begin{pmatrix} H_{n-2} & 0\\ S_{n-2} & 1\end{pmatrix} = \prod_{k=0}^{n-2}\begin{pmatrix} P_k & 0\\ R_k & 1\end{pmatrix}. \tag{4}$$

If U_n denotes the row vector $(f(n+1), f(n))$, equation (2) can be written as

$$(U_n, 1) = (U_0, 1)\begin{pmatrix} H_{n-1} & 0\\ S_{n-2} & 1\end{pmatrix}. \tag{5}$$

In case $r(x)=0$ for all x, the equation (1) is homogeneous, $R_k=0$ and $S_k=0$ for all k, and the solution is

$$U_n = U_0 H_{n-2}. \tag{6}$$

Moreover, each element of H_n is a solution of the homogeneous form of equation (1) *for special initial values.*

If $h(n)$, $k(n)$, and $s(n)$ denote the elements of the first column of the matrix product in equation (3), we get the standard form of the solution

$$f(x) = h(x)f(1) + k(x)f(0) + s(x) \tag{7}$$

in which $s(x)$ is called the particular solution.

The form of the particular solution* is of some interest and can be easily deduced from equation (4). Observe that

$$\begin{pmatrix} H_{n-1} & 0 \\ S_{n-1} & 1 \end{pmatrix} = \begin{pmatrix} H_{n-2} & 0 \\ S_{n-2} & 1 \end{pmatrix} \begin{pmatrix} P_{n-1} & 0 \\ R_{n-1} & 1 \end{pmatrix}, \tag{8}$$

so that

$$H_{n-1} = \prod_{k=0}^{n-1} P_k, \tag{9}$$

and

$$S_{n-1} = S_{n-2}P_{n-1} + R_{n-1}. \tag{10}$$

A consequence of (8) or (9) is that, if H_{n-2} is nonsingular,†

$$P_{n-1} = H_{n-2}^{-1} H_{n-1},$$

so that

$$S_{n-1} = S_{n-2}H_{n-2}^{-1}H_{n-1} + R_{n-1},$$

or

$$S_{n-1}H_{n-1}^{-1} - S_{n-2}H_{n-2}^{-1} = R_{n-1}H_{n-1}^{-1}.$$

This can be written in terms of the difference operator as

$$\Delta(S_{n-2}H_{n-2}^{-1}) = R_{n-1}H_{n-1}^{-1}.$$

Thus the particular solution can be derived by a summation process applied to an expression involving the nonhomogeneous term of equation (1) and the solutions of the homogeneous form of the equation. This same result can be derived by the method called variation of parameters.

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* This part of the note was added at the suggestion of the referee.

† H_n will be singular if, and only if, $q(k) = 0$ for $k \leqq n$.

REMARK ON A NOTE OF P. TURÁN

T. S. NANJUNDIAH, Mysore University

Using some properties of the Legendre polynomials, P. Turán [1] has given a proof of the combinatorial identity

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k}=\binom{n+k}{k}^2, \tag{1}$$

which occurred without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867 (from the author's summary as reported in the Mathematical Reviews, vol. 16, 1955, p. 13). Rewriting the sum in (1) in the reverse order and changing the notation slightly, (1) takes the form

$$\sum_{r=0}^{n}\binom{n}{r}^2\binom{\mu+r}{2n}=\binom{\mu}{n}^2, \qquad n=0,1,\cdots. \tag{2}$$

Now (2) is obviously a special case of the identity

$$\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n}{r}\binom{\mu+r}{m+n}=\binom{\mu}{m}\binom{\mu}{n}, \qquad m,n=0,1,\cdots. \tag{3}$$

Here we give a very simple and brief proof of (3) which involves just three applications of Vandermonde's identity.

Plainly we may take the summation in (3) over the range $0\leqq r\leqq n$. We have

$$\begin{aligned}\sum_{r=0}^{n}\binom{m}{r}\binom{n}{r}\binom{\mu+r}{m+n}&=\sum_{r=0}^{n}\binom{m}{r}\binom{n}{r}\sum_{s=0}^{r}\binom{r}{s}\binom{\mu}{m+n-s}\\&=\sum_{s=0}^{n}\binom{\mu}{m+n-s}\binom{m}{s}\sum_{r=s}^{n}\binom{m-s}{r-s}\binom{n}{n-r}\\&=\sum_{s=0}^{n}\binom{\mu}{m+n-s}\binom{m}{s}\binom{m+n-s}{n-s}\\&=\binom{\mu}{m}\sum_{s=0}^{n}\binom{m}{s}\binom{\mu-m}{n-s}=\binom{\mu}{m}\binom{\mu}{n},\end{aligned}$$

and the proof of (3) is complete. The same method of proof actually yields the more general identity

$$\sum_{r=0}^{n}\binom{m-\mu+\nu}{r}\binom{n+\mu-\nu}{n-r}\binom{\mu+r}{m+n}=\binom{\mu}{m}\binom{\nu}{n}, \qquad m,n=0,1,\cdots, \tag{4}$$

holding for arbitrary complex μ, ν.

Reference

1. P. Turán, On a problem in the history of Chinese mathematics, Mat. Lapok, vol. 5, 1954, pp. 1–6 (Hungarian, Russian and English summaries).

CONDITIONS FOR A POSITIVE-DEFINITE QUADRATIC FORM ESTABLISHED BY INDUCTION

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We give an inductive proof that the necessary and sufficient conditions for a quadratic form to be positive-definite are that the leading principal (abbreviated $l-p$) minors of the matrix of the form should be positive. Although A. Mirsky in *An Introduction to Linear Algebra* proves sufficiency by induction with effectively a condensation process, the present account may be of interest because it treats both sufficiency and necessity together and avoids much of the mechanical manipulation appearing in most textbook proofs. This is accomplished by the use of some simple properties of determinants and matrices and for these reference may be made to Chapters I and II of A. C. Aitken's *Determinants and Matrices*.

(i) *One variable.* Obviously $a_{11}x_1^2>0$ for all nonzero x_1 if and only if $a_{11}>0$. Here the $l-p$ minor is a_{11}, the sole element of the matrix, so that the theorem is true for this case.

(ii) *n variables.* Assume the theorem true for n variables x_i which can be considered as the elements of a column vector $\mathbf{x}$. Transposing, we write $\mathbf{x}' = [x_1 \cdots x_n]$ and take the matrix of the quadratic form as $\mathbf{A}=[a_{ij}]$ where, without loss of generality, $a_{ij}=a_{ji}$. i, j run 1 to n. The assumed necessary and sufficient conditions for positive-definiteness are that the n $l-p$ minors $|a_{rs}|$, obtained by letting r, s run 1 to 1, 2, $\cdots$ or n respectively, are all positive.

(iii) *$n+1$ variables.* Consider now the case of $n+1$ variables, denoting the extra variable as x_0 and the additional matrix elements as

$$a_{00} \text{ and } \mathbf{a}_0' = [a_{0i}] = [a_{i0}], \qquad i = 1 \text{ to } n,$$

so that the corresponding quadratic form is

$$T = [x_0\ \mathbf{x}'] \begin{bmatrix} a_{00} & \mathbf{a}_0' \\ \mathbf{a}_0 & \mathbf{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}.$$

The new matrix is

$$\mathbf{A}^+ = [a_{ij}], \qquad i, j = 0 \text{ to } n.$$

The matrices are partitioned conformably, so by multiplying out and noting symmetry we have

$$T = a_{00}x_0^2 + 2\mathbf{x}'\mathbf{a}_0x_0 + \mathbf{x}'\mathbf{A}\mathbf{x}.$$

Regarding this as a quadratic in x_0, we have $T>0$ for all values of the variables, exclusive of the case when all $n+1$ are zero, if and only if

$$(C_1) \qquad a_{00} > 0$$

and

(C_2) $$(\mathbf{x}'\mathbf{a}_0)^2 < a_{00}(\mathbf{x}'\mathbf{A}\mathbf{x}).$$

This latter condition is

$$\mathbf{x}'(a_{00}\mathbf{A} - \mathbf{a}_0\mathbf{a}_0')\mathbf{x} > 0,$$

involving a new quadratic form with matrix

$$\mathbf{B} = a_{00}\mathbf{A} - \mathbf{a}_0\mathbf{a}_0'.$$

Its elements $b_{ij}=a_{00}a_{ij}-a_{i0}a_{0j}$ are the 2×2 minors of $\mathbf{A}^+$ with a_{00} as leading pivot. A relation between the $l-p$ minors of $\mathbf{A}^+$ and $\mathbf{B}$ can be obtained as follows:*

If $\boldsymbol{\alpha}=-a^{-1}\mathbf{a}_0$ and $\mathbf{I}$ is the $n\times n$ unit matrix,

$$\begin{bmatrix} 1 & 0 \\ \boldsymbol{\alpha} & \mathbf{I} \end{bmatrix}\begin{bmatrix} a_{00} & \mathbf{a}_0' \\ \mathbf{a}_0 & \mathbf{A} \end{bmatrix}\begin{bmatrix} 1 & \boldsymbol{\alpha}' \\ 0 & \mathbf{I} \end{bmatrix} = \begin{bmatrix} a_{00} & 0 \\ 0 & a_{00}^{-1}\mathbf{B} \end{bmatrix}.$$

Since the prefactor merely adds multiples of the first row of $\mathbf{A}^+$ to subsequent rows and the postfactor operates on columns similarly, any leading determinant of $\mathbf{A}^+$ and of this product have the same value. Thus for the $(r+1)$th $l-p$ minor

$$(1) \qquad \left| a_{00}a_{11}\cdots a_{rr} \right| = a_{00}^{1-r}\left| b_{11}b_{22}\cdots b_{rr} \right|, \qquad r = 1 \text{ to } n,$$

a relationship often used in evaluating determinants by condensation.

Because we are assuming the results expressed in (ii) apply to $n\times n$ matrices, condition C_2 is effectively that all the $l-p$ minors of $\mathbf{B}$ are positive. If also $a_{00}>0(C_1)$, it follows from (1) that the $l-p$ minors of $\mathbf{A}^+$ of orders 2 to $n+1$ are positive and a_{00} is its first minor.

Therefore C_1 and C_2 together are equivalent to the condition that the $n+1$ $l-p$ minors of $\mathbf{A}^+$ are all positive. Conversely if the last statement is true, it is immediately apparent that C_1 and C_2 hold.

The validity of the first case has been noted, the necessity and sufficiency of the conditions given in the theorem have been shown to apply to the $n+1$th case if they apply to the nth case and the inductive proof is thus complete.

* I wish to thank the referee for suggesting this form of demonstration of (1) for inclusion here.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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FOR THE HOMOGENEOUS DIFFERENTIAL EQUATION $M(x, y)dx+N(x, y)dy=0$, $y=vx$ or $x=vy$?

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In most elementary differential equations texts when the homogeneous equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

is considered, it is shown that either substitution $y=vx$ or $x=vy$ will always transform the equation to one where the variables are separable. Usually the remark is made that either substitution will work. Most often, however, it is not clear which substitution will lead to the simpler integration on v, and it is necessary to study both resulting integrands to decide this. Thus, the equation is almost solved twice; the following device indicates how this process can be substantially shortened.

Write (1) in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}. \tag{2}$$

Since M and N are homogeneous functions of the same degree, (2) can be put into the form

$$\frac{dy}{dx} = -f\left(\frac{y}{x}\right) = -g\left(\frac{x}{y}\right). \tag{3}$$

Setting $y/x=v$, and $x/y=u$ for convenience, (3) yields

$$\frac{dx}{x} + \frac{dv}{v + f(v)} = 0, \tag{4}$$

$$\frac{dy}{y} + \frac{du}{u + \dfrac{1}{g(u)}} = 0. \tag{5}$$

The question is now which is simpler to integrate, (4) or (5). To decide this, use the facts that $f(v)=g(u)$ and $v=1/u$. Then (5) becomes

$$\frac{dy}{y} + \frac{du}{u + \dfrac{1}{f(1/u)}} = 0. \tag{6}$$

The problem has been reduced to finding which is easier to evaluate,

$$\int \frac{dv}{v + f(v)}, \tag{7}$$

$$\int \frac{dv}{v + \dfrac{1}{f(1/v)}}. \tag{8}$$

This analysis leads to a convenient rule for solving such a differential equation. Except for those special cases in which it is evident that the substitution $x = vy$ will prove simpler, set $y = vx$ and consider equation (4) which results. In particular, from (7) which occurs in the integration of equation (4), form (8). If (7) is simpler to evaluate than (8), the solution of the equation will be

$$\ln |x| + \int \frac{dv}{v + f(v)} = c \quad \text{where} \quad v = \frac{y}{x}. \tag{9}$$

If (8) is simpler to find than (7), the solution will be

$$\ln |y| + \int \frac{dv}{v + \dfrac{1}{f(1/v)}} = c \quad \text{where} \quad v = \frac{x}{y}. \tag{10}$$

An example will illustrate the technique.

Solve:

$$(2xy + 3y^2)dx + (5y^2 - x^2)dy = 0. \tag{11}$$

Since it is not clear which substitution will result in the simpler integration on v, set $y = vx$, and (11) becomes

$$\int \frac{dx}{x} + \int \frac{5v^2 - 1}{5v^3 + 3v^2 + v}\, dv = c. \tag{12}$$

Although this second integral can be evaluated by the method of partial fractions, a look at the integral that would have resulted from the substitution $x = vy$ is justified. To obtain this write this integral in the form (7). Thus

$$\int \frac{5v^2 - 1}{5v^3 + 3v^2 + v}\, dv = \int \frac{dv}{v + \dfrac{3v^2 + 2v}{5v^2 - 1}}. \tag{13}$$

Here

$$f(v) = \frac{3v^2 + 2v}{5v^2 - 1}$$

and

$$\frac{1}{f(1/v)} = \frac{1}{\dfrac{\dfrac{3}{v^2}+\dfrac{2}{v}}{\dfrac{5}{v^2}-1}} = \frac{5-v^2}{3+2v}.$$

Now (8) becomes

$$\int \frac{(2v+3)dv}{v^2+3v+5} \tag{14}$$

which is immediately integrable; now using (14) in (10), the solution to equation (11) reduces to

$$x^2+3xy+5y^2 = cy. \tag{15}$$

The same device could be used to replace (8) by (7) when starting with the substitution $x=vy$ when it is desired to see the integral on v that would have resulted from $y=vx$. That is, from (8) occurring in the solution (10) find $[f(1/v)]^{-1}$, hence $f(1/v)$ and therefore $f(v)$, and thus form (7). If this is simpler to find than (8), (9) gives the complete solution.

A PROJECTIVE DEFINITION OF DOUBLE RATIO

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Most textbooks in projective geometry introduce the idea of double ratio of four points on a line by means of a metric definition in terms of distances between the points and then proceed to prove that the double ratio is a projective invariant. The following definition, although not new, does not seem to be very well known.

With a projective system of coordinates in n-space, let $(\alpha x)\equiv\alpha_0x_0+\cdots+\alpha_nx_n=0$ and $(\beta x)=0$ be the equations of two hyperplanes or primes and let $a\equiv(a_0, \cdots, a_n)$ and $b\equiv(b_0, \cdots, b_n)$ be two points not incident with either of the two primes.

DEFINITION. *The double ratio D of the two points a and b with respect to the two primes $(\alpha x)=0$ and $(\beta x)=0$ is*

$$D = \frac{(\alpha a)(\beta b)}{(\alpha b)(\beta a)} \tag{1}$$

where $(\alpha a)\equiv(\alpha_1a_0+\cdots+\alpha_na_n)$, etc. The expression D is obviously an invariant under linear transformation since each of its factors is. Moreover it is independent of any constant of proportionality in the coordinates of the points or the coefficients of the primes. It is therefore a geometric invariant of the points and primes.

In (1) let D, α_i, β_i, b_i $(i=0, 1, \cdots, n)$ be constants while a_i is replaced by a variable x_i. Then (1) becomes a linear equation, obviously satisfied by the point a and the base of the pencil of primes determined by $(\alpha x)=0$ and $(\beta x)=0$. Call this prime $(\gamma x)=0$ and let $(\delta x)=0$ be the equation of the similar prime on b and the base of the same pencil of primes. Then D is the double ratio of the four primes α, β, γ, δ and a and b are any two points of γ and δ respectively.

Dually, let D, a_i, b_i, β_i be constants and let α_i be replaced by a variable μ_i. Then (1) becomes the prime equation of the point c in which the prime α meets the line ab. Similarly let the point d be the meet of the prime β and the line ab. Hence D is the double ratio of the four points a, b, c, d on a line.

Now if α, β, γ, and δ are primes of a parallel pencil, equivalent projectively to any pencil of primes, and a and b are points of γ and δ respectively, then the quantities (αa), *etc.*, are proportional to the actual distances from the primes to the points and D satisfies the usual metric definition of double ratio.

In order to compute the double ratio of four primes of a pencil we may choose any one of them for α and any other one for β. Then a is chosen as any point of either of the remaining primes and b is any point of the other. Hence we obtain the 24 possible double ratios. If D is represented by the symbol $(\alpha\beta, ab)$ then, obviously $(\alpha\beta, ab)=(\alpha\beta, ba)=(ab, \alpha\beta)=(ba, \alpha\beta)$ since $(\alpha a)\equiv(a\alpha)$, *etc.*

In particular, in a linear space S_1 let two points be given by their coordinates: $a(a_0, a_1)$, $b(b_0, b_1)$ and two by their equations: $c(\gamma x)=0$, $d(\delta x)=0$. Then the double ratio

$$(ab, cd) = \frac{(\gamma a)(\delta b)}{(\gamma b)(\delta a)}.$$

But if $\gamma_0 x_0+\gamma_1 x_1=0$ then $x_0:x_1=-\gamma_1:\gamma_0$ and if c has the coordinates (c_0, c_1), then $c_0:c_1=-\gamma_1:\gamma_0$. Similarly if d has the coordinates (d_0, d_1), then $d_0:d_1 = -\delta_1:\delta_0$ and D may be written

$$(ab, cd) = \frac{(c_1a_0 - c_0a_1)(d_1b_0 - d_0b_1)}{(c_1b_0 - c_0b_1)(d_1a_0 - d_0a_1)} = \frac{|ac||bd|}{|ad||bc|}.$$

Similarly the double ratio of four points of a line given by their equations $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ is

$$(\alpha\beta, \gamma\delta) = \frac{|\alpha\gamma||\beta\delta|}{|\alpha\delta||\beta\gamma|}.$$

FINDING THE RATE IN A CONTINUOUS ANNUITY

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In this note some formulas are presented for very accurate determination of the rate of interest in a continuous annuity. The present value of a continuous

annuity is given by the formula,

$$B = Ra = \frac{R(1 - e^{-n\delta})}{\delta} \tag{1}$$

where R = periodic payment, n = number of periodic payments, B = amount of loan, δ = rate of interest (to be determined).

Now formula (1) can be written in the form

$$B = \frac{I}{\dfrac{x}{1 - e^{-x}} - 1} \tag{2}$$

where $I = Rn - B$, the cost of the loan, and $x = n\delta$.

The right member of (2) may be expanded in the following series:

$$B = \frac{2I}{x}\left(1 - \frac{x}{6} + \frac{x^2}{36} - \frac{x^3}{540} - \frac{x^4}{6480} \cdots\right). \tag{3}$$

We now invert (3) and obtain

$$x = 2M\left(1 - \frac{M}{3} + \frac{2M^2}{9} - \frac{22M^3}{135} - \frac{7M^4}{135} \cdots\right) \tag{4}$$

where $M = I/B$.

Now (4) is approximately equal to

$$x' = \frac{2M}{\sqrt[3]{1 + M}} = \frac{2I}{\sqrt[3]{B^2Rn}} \tag{5}$$

with an error less than $+8M^4/405$, the result being a little too small (*i.e.*, $x - x' \doteq +8M^4/405$). Formula (5) is an analog of Schurig's formula.

Also (4) may be expressed as a continued fraction:

$$x = 2M\left(\cfrac{1}{1 + \cfrac{\frac{M}{3}}{1 + \cfrac{\frac{M}{3}}{1 + \cfrac{\frac{2M}{15}}{\cdots}}}}\right). \tag{6}$$

By considering convergents, we obtain some very excellent approximations to the rate [1]

$$x_1 = \frac{2M}{1+\frac{M}{3}}\left(\text{error} < +\frac{2M^3}{9}\right), \tag{7}$$

$$x_2 = 2M\left[\frac{3+M}{3+2M}\right]\left(\text{error} < -\frac{4M^4}{135}\right), \tag{8}$$

$$x_3 = \frac{6M(15+7M)}{45+36M+2M^2}\left(\text{error} < -\frac{698}{2025}M^5\right). \tag{9}$$

A very good approximation is the average of (5) and (8),

$$x \doteq \frac{x'+x_2}{2}\left(\text{error} < -\frac{2M^4}{405}\right). \tag{10}$$

δ can be found by recalling that $\delta = x/n$. It should be noted that (7) gives too small a result, (8) gives a better result, but too large, while (9) gives a still better result, but slightly large.

Example. Let $B = 1{,}906{,}083.04$ and $R = 100{,}000$ then $I = 493{,}916.94$ and $M = .2591266747$. By computation with a desk calculator, we find

$$x_1 = .477048046, \qquad \delta_1 = .019877 \text{ (error} + .000123),$$
$$x_2 = .480082912, \qquad \delta_2 = .02000345 \text{ (error} - .0000034),$$
$$x_3 = .479988792, \qquad \delta_3 = .019999533 \text{ (error} + .000000467),$$
$$x' = .4799397, \qquad \delta' = .0199975 \text{ (error} + .0000025).$$

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PROOF OF THE MEAN VALUE THEOREM

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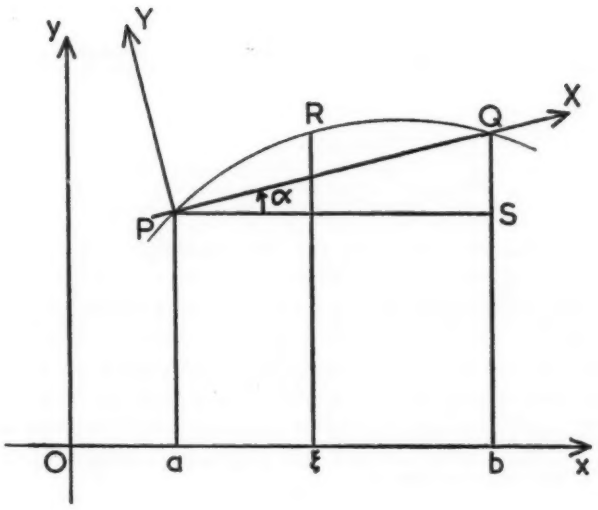
A proof is given here of the mean value theorem, based on a translation and rotation of axes, which is believed to be new.

The theorem in its usual form is as follows: Let $y = f(x)$ be continuous for $a \leqq x \leqq b$ with a derivative, $f'(x)$, such that $a < x < b$. Then, there is at least one number ξ such that $f(b) - f(a) = (b-a)f'(\xi)$, where $a < \xi < b$.

In the accompanying figure let PRQ be a portion of the graph of $y = f(x)$ and let chord PQ make angle α with PS, where PS is parallel to the x-axis. Then,

(1) $$\tan \alpha = \frac{SQ}{PS} = \frac{f(b) - f(a)}{b - a}.$$



Refer $y=f(x)$ to a new system of axes (X, Y) with origin at $P[a, f(a)]$ and with α as the angle through which the (x, y) system is rotated to bring it into coincidence with the (X, Y) system. We then have

$$X \cos \alpha - Y \sin \alpha = x - a, \qquad X \sin \alpha + Y \cos \alpha = y - f(a),$$

from which we find

(2)
$$X = \frac{\begin{vmatrix} x - a & -\sin \alpha \\ y - f(a) & \cos \alpha \end{vmatrix}}{\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}} = [y - f(a)] \sin \alpha + (x - a) \cos \alpha,$$

$$Y = \frac{\begin{vmatrix} \cos \alpha & x - a \\ \sin \alpha & y - f(a) \end{vmatrix}}{\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix}} = [y - f(a)] \cos \alpha - (x - a) \sin \alpha.$$

Then, if $Y=F(X)$ is the equation of the curve with respect to the new system of axes, we have

$$(3)\qquad \begin{aligned} F(X) &= [f(x) - f(a)]\cos\alpha - (x - a)\sin\alpha, \\ X &= [f(x) - f(a)]\sin\alpha + (x - a)\cos\alpha, \end{aligned}$$

from which it follows that

$$\frac{dF(X)}{dx} = f'(x)\cos\alpha - \sin\alpha, \qquad \frac{dX}{dx} = f'(x)\sin\alpha + \cos\alpha.$$

Hence,

$$F'(X) = \frac{dF(X)}{dX} = \frac{f'(x)\cos\alpha - \sin\alpha}{f'(x)\sin\alpha + \cos\alpha}.$$

From (3), when $x=a$, $F(X)=0$, and since from (1),

$$\sin\alpha = \frac{f(b) - f(a)}{\sqrt{[f(b) - f(a)]^2 + (b-a)^2}}, \qquad \cos\alpha = \frac{b - a}{\sqrt{[f(b) - f(a)]^2 + (b - a)^2}},$$

we also have $F(X)=0$ when $x=b$. Hence, by Rolle's theorem, $F'(X)$ must vanish for some value of x between a and b.* Let this value be $x=\xi$. We then have

$$\frac{f'(\xi)\cos\alpha - \sin\alpha}{f'(\xi)\sin\alpha + \cos\alpha} = 0 \quad \text{or} \quad f'(\xi)\cos\alpha - \sin\alpha = 0.$$

It follows from this last equation that $\tan\alpha = f'(\xi)$.

The result of substituting this in (1) is $f(b) - f(a) = (b-a)f'(\xi)$.

* In exceptional cases where $F(X)$ does not satisfy all the conditions of Rolle's theorem, the proof given is not valid.

ON THE DERIVATIVE OF x^c

S. LEADER, Rutgers University

For real constants a and c with $a>0$, the differentiation formula

$$(1)\qquad Dx^c\big]_{x=a} = ca^{c-1}$$

can be obtained directly from the special case in which $a=1$ and $c>0$:

$$(2)\qquad \lim_{x\to 1}\frac{x^c - 1}{x - 1} = c.$$

To derive (1) from (2) we have only to use the identities

$$(3)\qquad \frac{x^c - a^c}{x - a} = \left(\frac{w^c - 1}{w - 1}\right)a^{c-1} = -\,w^c\left(\frac{w^{-c} - 1}{w - 1}\right)a^{c-1}$$

where $w=x/a$, noting that w approaches 1 as x approaches a. Thus, we need only prove (2).

For the case $c=n$, a positive integer, (2) follows from the identity

$$\frac{x^n-1}{x-1}=1+x+x^2+x^3+\cdots+x^{n-1}. \tag{4}$$

For $c=m/n$, a positive rational, we have, setting $y=x^{1/n}$,

$$\frac{x^{m/n}-1}{x-1}=\frac{\dfrac{y^m-1}{y-1}}{\dfrac{y^n-1}{y-1}} \tag{5}$$

which gives (2) *via* the preceding case.

For arbitrary $c>0$ consider any rationals p and q such that $0<p<c<q$. Then for $0<x<1$ we have $x^q<x^c<x^p$ so $x^q-1<x^c-1<x^p-1$. Since $x-1<0$,

$$\frac{x^p-1}{x-1}<\frac{x^c-1}{x-1}<\frac{x^q-1}{x-1}. \tag{6}$$

Now (6) also holds for $x>1$, for then $x^p<x^c<x^q$ and $x-1>0$. Letting x approach 1 in (6) we obtain

$$p\leqq\lim_{x\to 1}\frac{x^c-1}{x-1}\leqq q. \tag{7}$$

Since p and q may be arbitrarily close to c, (7) gives (2).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1316. *Proposed by R. C. Buck, University of Wisconsin*

Let S be a set of elements with an associative multiplication. Suppose that S has a special element u with the property that u is a left and a right divisor of every element in S. Does S have to possess a unit?

E 1317. *Proposed by N. A. Court, University of Oklahoma*

Let $(T)=DABC$ be a tetrahedron and let (M) be a sphere having its center M on the axis of the circle ABC. If one and only one point is taken in each

pair of points determined by (M) on the edges of (T) issued from the vertex D, show that the $2^3=8$ planes thus determined are cut by the plane ABC along four pairs of isotomic transversals of the triangle ABC.

E 1318. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories, Inc.*

A. A. Mullin has proved (this MONTHLY [1957, p. 669]) that

$$\Phi(n) = \sum_{r=0}^{n} {}_nP_r = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \sim (n!)e$$

for large n, where $\sim$ denotes asymptotic equality. Prove that

$$\Phi(n) = [(n!)e], \qquad n \geqq 1,$$

where $[x]$ denotes the largest integer not greater than x.

E 1319. *Proposed by S. W. Golomb, California Institute of Technology*

A quiz contestant selects a category containing n questions, k of which are too difficult for him. The questions are selected from the category at random, and the contestant continues answering until he misses a question. What is the probability that he will miss on the ath question?

E 1320. *Proposed by G. S. Stoller, Polytechnic Institute of Brooklyn*

(a) Let $P(x)$ be any polynomial in x with integral coefficients. Prove that there exists an infinite number of integers t such that $P(t)$, $P(t+1)$, $\cdots$, $P(t+m)$ are all composite for any given positive integer m.

(b) Let $P(x)=x^2+1$. Find a value of t satisfying part (a) for $m=5$.

SOLUTIONS

Two Equal Sums of n Distinct Squares

E 1286 [1957, 670]. *Proposed by F. S. Stancliff, Springfield, Ohio*

Establish the identity

$$\sum_{r=0}^{n-2} (2^r - 1)^2 + [3(2^{n-1}) - 1]^2 = \sum_{r=2}^{n} (2^r - 1)^2 + (2^n - 4)^2,$$

thus obtaining a general method for finding two equal sums of n distinct squares.

Solution by C. F. Pinzka, University of Cincinnati. The identity is established by noting that

$$\sum_{r=2}^{n} (2^r + 1)^2 - \sum_{r=0}^{n-2} (2^r - 1)^2 = \sum_{r=0}^{n-2} [(4 \cdot 2^r + 1)^2 - (2^r - 1)^2]$$

$$= 5 \sum_{r=0}^{n-2} (3 \cdot 2^{2r} + 2^{r+1}) = 5(2^{2n-2} - 1 + 2^n - 2)$$

$$= (5 \cdot 2^{n-1} - 5)(2^{n-1} + 3) = [3 \cdot 2^{n-1} - 1]^2 - (2^n - 4)^2.$$

Also solved by D. S. Adorno, W. A. Al-Salam, Winifred Asprey, D. A. Breault, D. R. Brillinger, P. L. Chessin, W. J. Cody, A. E. Danese, Underwood Dudley, E. S. Eby, W. V. Gamzon, Lawrence Glasser, Cornelius Groenewoud, Emil Grosswald, Leonard Hauer, Vern Hoggatt, Edgar Karst, M. A. Kirchberg, J. D. E. Konhauser, Joe Lipman, D. C. B. Marsh, J. B. Muskat, Tim Nolan, F. D. Parker, M. J. Pascual, Susan Pyeatt, L. A. Ringenberg, D. A. Robinson, R. E. Shafer, Ya'akov Shima, Paul Slepian, Peter Treuenfels, C. W. Trigg, Dale Woods, and David Zeitlin. Late solutions by E. F. Allen, D. W. Bailey, Julian Braun, Eugene Famolari, Jr., R. H. Hou, A. R. Hyde, D. L. Muench, and Dmitri Thoro.

Equilateral Triangles on the Sides of a Given Triangle

E 1287 [1957, 671]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the radical center P of the circles inscribed in the equilateral triangles BCA', CAB', ABC' constructed exteriorly (or interiorly) on the sides of a triangle ABC having centroid G divides the distance from G to the point of concurrence Q of the lines AA', BB', CC' in the ratio $GP/GQ=1/4$.

Solution by H. E. Fettis, Dayton, Ohio. The proposition may be established with the aid of the following lemma:

The segments of the line joining the midpoints of any two sides of the triangle ABC intercepted by the circles inscribed in the corresponding equilateral triangles are equal. Let the midpoints of the sides be designated by L, M, N and the centers of the equilateral triangles by D, E, F. Then, if MN intersects circle (E) again in R and circle (F) in S,

$$EM/RM = AC/2h, \qquad FN/NS = AB/2h,$$

where h is the altitude drawn from A. Hence

$$(AC)(RM)/EM = (AB)(NS)/FN.$$

But, since the triangles are equilateral, $AC/EM=AB/FN$, so that $RM=NS$.

Proceeding to the main problem, it is known that DEF is equilateral, with sides mutually perpendicular to AA', BB', CC'. Consider, now, the parallelogram $MNVW$, where V and W are the remaining points of contact of circle (D). If the common midpoint T of VW and LA' is joined to the midpoint K of LA, intersecting GQ at P, this join will be parallel to AA'. Also, since VN and WM are parallel to AA', K must also be the midpoint of MN. Further, this line which, by construction, divides the median segment GA at K in the ratio 1/4 must also divide GQ at P in the ratio 1/4. Now, by virtue of the lemma, K has equal powers with respect to the circles (E) and (F), and since KT is perpendicular to EF, it is the radical axis of these circles. Thus the radical axis of any pair of circles divides GQ at P in the ratio 1/4, and therefore P is the radical center of the circles (D), (E), (F).

An analogous proof may be given for the case where the triangles are described internally.

Also solved by W. B. Carver, J. W. Clawson, Roscoe Woods, and the proposer.

Woods gave a synthetic proof based on the lemma: *Let O be the circumcenter of a triangle ABC and let P be any point in the plane of the triangle. The radical center M of the circles whose centers are A, B, C and whose radii are $k(AP)$, $k(BP)$, $k(CP)$, where k is a parameter, lies on the line OP and such that $OM/OP=k^2$.* Applying the lemma to triangle DEF, letting Q play the role of P in the lemma, and taking $k=1/2$, the theorem readily follows.

Carver employed the method of conjugate coordinates. Late solutions by E. F. Allen and Josef Langr.

Odd Binomial Coefficients

E 1288 [1957, 671]. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this MONTHLY [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

I. *Solution by T. R. Hatcher and J. A. Riley, Parke Mathematical Laboratories, Carlisle, Mass.*

Let h and n be positive integers with $h<n$. We define the *binary length* of n, $L(n)$, to be the number of ones in the binary representation of n, and the *binary capacity* of n, $C(n)$, to be the exponent of the highest power of two which divides n. We say "h is contained in n," written $h\subset n$, if when h has a one in a certain binary place, n also has a one in the corresponding binary place; that is, the binary representation of h can be obtained from that of n by changing ones to zeros.

The following properties are easily proved:

(1) $C(n)$ is the number of terminating zeros in the binary representation of n.

(2) $C(n)=0$ if and only if n is odd.

(3) $C(ab)=C(a)+C(b)$, $C(a/b)=C(a)-C(b)$.

(4) $L(n)=L(h)+L(n-h)$ if and only if $h\subset n$.

(5) $C(n)=1+L(n-1)-L(n)$.

(6) $C(n!)=n-L(n)$.

(7) $C\binom{n}{h}=L(h)+L(n-h)-L(n)$.

The corollary of the following theorem gives the solution.

THEOREM. $\binom{n}{h}$ *is odd if and only if* $h\subset n$.

Proof. If $\binom{n}{h}$ is odd, $C\binom{n}{h}=0$ and by (7) $L(n)=L(h)+L(n-h)$. Thus, by (4), $h\subset n$. Conversely, if $h\subset n$, then $L(n)=L(h)+L(n-h)$ and $C\binom{n}{h}=0$.

COROLLARY. *The number of integers h such that $\binom{n}{h}$ is odd is $2^{L(n)}$.*

Proof. The number of integers h with $h\subset n$ and $L(h)=j$ is $\binom{L(n)}{j}$. Thus the number of integers h for which $\binom{n}{h}$ is odd is simply

$$\sum_{j=0}^{L(n)}\binom{L(n)}{j}=2^{L(n)}.$$

II. *Remarks by Leo Moser, University of Alberta.* Problem E 1288 is a special case of 4723 [1957, 116]. The solution of that problem is the following:

If $n=a_0+a_1p+a_2p^2+\cdots+a_kp^k$, $0\leqq a_i<p$, $i=0, 1, 2, \cdots, k$, then the number of solutions of $(\binom{n}{r}, p)=1$, $r=0, 1, \cdots, n$, is $\prod_{i=0}^{k}(a_i+1)$.

The result in E 1288 is contained in J. W. L. Glaisher, "On the residue of a binomial coefficient with respect to a prime modulus," *Quarterly Journal of Mathematics*, vol. 30, 1899, pp. 150–156. More recently a proof was given by J. B. Roberts, "On binomial coefficient residues," *Canadian Journal of Mathematics*, vol. 9, 1957, pp. 363–370.

Also solved by D. R. Brillinger, Leonard Carlitz, Joe Lipman, D. C. B. Marsh, Paul Schillo, and the proposer.

The Fermat Equation

E 1289 [1957, 671]. *Proposed by Marvin Shinbrot, National Advisory Committee for Aeronautics, Moffett Field, California*

Show that the Fermat equation $x^n+y^n=z^n$ has no nontrivial solution in integers for $n>2$ if $z<2^{1/n}/(2^{1/n}-1)$.

Solution by G. S. Stoller, Polytechnic Institute of Brooklyn. Suppose there is a solution in integers of $x^n+y^n=z^n$ for $n>2$ and integral. Then, since $0<x<z$ and $0<y<z$, $z^n\leqq 2(z-1)^n$, or $z\geqq 2^{1/n}/(2^{1/n}-1)$.

Also solved by Leonard Carlitz, M. A. Kirchberg, D. C. B. Marsh, L. F. Meyers, J. B. Muskat, R. E. Shafer, and the proposer. Late solution by Walter Leighton and M. C. Sholander (jointly).

Inverse Tangents

E 1290 [1957, 671]. *Proposed by F. E. Shafer, University of California Radiation Laboratory, Livermore, California*

If $|n\tan^{-1}x|\leqq\pi/2$, show that

$$n\tan^{-1}x=\tan^{-1}\frac{\operatorname{Im}(1+ix)^n}{\operatorname{Re}(1+ix)^n},$$

where $\operatorname{Im}f(z)$ denotes the imaginary part of $f(z)$ and $\operatorname{Re}f(z)$ denotes the real part of $f(z)$.

Solution by W. A. Al-Salam, Duke University. Consider the complex number $z=1+ix$. We have $\arg z=\tan^{-1}x$. Hence it follows that $\arg z^n=n\tan^{-1}x$. Now, since $|n\tan^{-1}x|\leqq\pi/2$, we have the desired result.

Also solved by D. S. Adorno, R. J. Arguello, D. A. Breault, Tien Chi Chen and Bryant Tuckerman (jointly), P. L. Chessin, Underwood Dudley, E. S. Eby, Lawrence Glasser, Cornelius Groenewoud, Juris Hartmanis, Vern Hoggatt, J. R. Holdsworth, M. A. Kirchberg, M. S. Klamkin, J. D. E. Konhauser, Joe Lipman, D. C. B. Marsh, Morris Morduchow, C. S. Ogilvy, M. J. Pascual, C. F. Pinzka, Michael Rosen, E. M. Scheuer, Nathan Schwid, Ya'akov Shima, Arnold Singer, Paul Slepian, Peter Treuenfels, Chih-yi Wang, R. H. Wilson, Jr., Dale Woods, David Zeitlin, and the proposer. Late solutions by Edward Barbeau, Julian Braun, A. E. Danese, R. H. Hou, and Vencil Skarda.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4788. *Proposed by D. J. Newman, AVCO Research, Lawrence, Mass.*

Let $f(z)$ be an entire function and consider the two statements: (1) If f, f', f'' never vanish, then $f(z) = e^{az+b}$. (2) If f, f', f'' each miss a value, then $f(z) = a + be^{cz}$. (1) is a known theorem. (2) is apparently stronger than (1). Show, however, that (2) is an elementary consequence of (1).

4789. *Proposed by Kurt Mahler and P. M. Cohn, the University, Manchester, England*

Let A be an associative algebra over a field of characteristic zero. If $(x+y)^n = \sum_r \binom{n}{r} x^r y^{n-r}$ for some $n \geqq 2$ and for all x and y in A, is it true that $(x+y)^{n+1} = \sum_r \binom{n+1}{r} x^r y^{n+1-r}$ for all x and y in A?

4790. *Proposed by Leonard Carlitz, Duke University*

Let p be a fixed prime >2 and put $\psi(x) = (x/p)$, the Legendre symbol. Evaluate the sum

$$N_r(a) = \sum_{x_1, \cdots, x_r = 1}^{p-1} \psi\{x_1 \cdots x_r (a - x_1 - \cdots - x_r)\}.$$

4791. *Proposed by R. C. Lyness, Preston, England*

If α, β, γ are real and $\alpha^3 + \beta^3 + \gamma^3 = 0$, prove

$$[\sum (\beta - \gamma)^2][\sum \alpha^4] \geqq [\sum \alpha^2]^3.$$

4792. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research, Lawrence, Mass.*

Show that the following operators are reducible:

(1) $x^n D^{2n}$, (2) $x^{2n} D^n$,

and thus solve the differential equations

(1a) $[x^n D^{2n} - \lambda]y = 0$, (2a) $[x^{2n} D^n - \lambda]y = 0$.

SOLUTIONS

Editorial Note. A. Makowski of Warsaw, Poland, has interested himself in the list of unsolved

MONTHLY Problems in the recent *Otto Dunkel Memorial Problem Book* (August–September 1957, Part II). We are indebted to him for the following comments and solutions. Similar assistance from other readers will be most welcome.

2798 [1919, 458]. *Proposed by S. A. Corey.* Prove that every positive integer is equal to the sum of at most four squares.

Comment by Andrzej Makowski, Warsaw, Poland. This is a well-known theorem of Lagrange (or Bachet). See, *e.g.*, Nagell, *Introduction to Number Theory*, New York, 1951, pp. 191–195.

2935 [1921, 467]. [Proposed in *Gazeta Matematică*, April 1921]. Find the integral solutions of the equation $x!+1=y^2$.

Comment by Andrzej Makowski, Warsaw, Poland. This is equivalent to Problem E 534 [1943, 261; 1950, 557; 1951, 193].

3057 [1924, 101]. *Proposed by A. A. Bennett.* Euler in a letter to Stirling stated (without any hint as to the method of proof) that $\pi/4$ might be represented as an infinite product where the nth factor is the quotient of the nth odd prime when divided by the integral multiple of 4 nearest to it. Prove that

$$\frac{\pi}{4}=\frac{3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot\cdots}{4\cdot4\cdot8\cdot12\cdot12\cdot16\cdot\cdots}.$$

Comment by Andrzej Makowski, Warsaw, Poland. This result was proved by E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2nd ed., 1953, Bd. 1, s. 449.

3133 [1925, 261]. *Proposed by A. A. Bennett.* Show that in every set containing but a finite number of elements and admitting an associative rule of multiplication, there must be an idempotent element, that is, one which is equal to its square. Show that this does not continue to hold if the set contains an infinite number of elements.

Solution by Andrzej Makowski, Warsaw, Poland. Let A be a set with the given properties, and let a be an element of A. By the associative rule, for every choice of the positive integers i, j, the equality $a^ia^j=a^{i+j}$ holds. Since A is finite, there exist positive integers n, k such that $a^{n+k}=a^n$. From this, we obtain $a^{n+sk}=a^n$ $(s=1, 2, \cdots)$, whence for $s=2n$, $a^{n(1+2k)}=a^n$. Finally, upon multiplication of both members by $a^{(2k-1)n}$, we have $(a^{2kn})^2=a^{2kn}$, so that a^{2kn} is an idempotent element.

If A is the set of integers $\geqq 2$ and multiplication is the ordinary multiplication of integers, then an idempotent element does not exist.

3951 [1940, 182]. *Proposed by V. V. Johnston.* Does the equation $m^3+3m^2+2m=2n^3+3n^2+n$ admit positive integral solutions in m and n other than $m=n=1$?

Comment by Andrzej Makowski, Warsaw, Poland. E. B. Escott (*L'Inter-*

médiaire des Math., vol. 8, 1901, p. 249) asked if the equation $n(n+1)(n+2)=m(m+1)(2m+1)$ has integral solutions.

Equations $m^3+3m^2+2m=2n^3+3n^2+n$ and $(m+1)^3-n^3-(n+1)^3=m-2n$ are equivalent. The last equation has no integral solution with $1<m<3164$, $1<n<3164$, $|2m-n|\leqq 100$. This follows immediately upon inspection of the tables of solutions of the equation $x^3+y^3+z^3=n$ which are given by Miller and Woollett (*J. London Math. Soc.*, vol. 30, 1955, pp. 101–110).

Vector-matrix Differential Equation, a Stability Theorem

4745 [1957, 437]. *Proposed by Marvin Marcus, National Bureau of Standards and the University of British Columbia*

Let $A(t)$ be a complex n-square matrix function, continuous on $[0, \infty)$. Let A^* be the conjugate transpose of A and let $X(t)$ be the real trace of $A(t)$. Assume that

$$\text{(i)} \quad A(t)+A^*(t)\geqq 0, \qquad 0\leqq t<\infty; \qquad \text{(ii)} \quad \limsup_{t\to\infty}\int_0^t X(s)ds<\infty.$$

Show that all solutions of the linear vector-matrix differential equation $\dot{y}=A(t)y$ are bounded on $[0, \infty)$.

Solution by Leopold Flatto, Reeves Instrument Corporation, New York City. $A+A^*\geqq 0$ is assumed to mean that the characteristic roots of $(A+A^*)$ are nonnegative. Let y be an n-dimensional vector, A an $n\times n$ matrix, and let $|y|^2=|y_1|^2+\cdots+|y_n|^2$. We have $dy/dt=Ay$, $dy^*/dt=y^*A^*$. Now

$$\frac{d(|y|^2)}{dt}=\frac{d(y^*y)}{dt}=y^*\frac{dy}{dt}+\frac{dy^*}{dt}y=y^*Ay+y^*A^*y=y^*(A+A^*)y.$$

Since $A(t)+A^*(t)$ is Hermitian, there exists a unitary matrix $U(t)$ which diagonalizes $A+A^*$. Let $y=Uz$; then

$$y^*(A+A^*)y=z^*U^*(A+A^*)Uz=\sum_{i=1}^n \lambda_i|z_i|^2,$$

where the λ_i are characteristic values of $A+A^*$. Hence

$$\frac{d(|y|^2)}{dt}=\sum_{i=1}^n \lambda_i|z_i|^2\leqq\sum_{i=1}^n \lambda_i(|z_1|^2+\cdots+|z_n|^2).$$

(The last inequality follows from the assumption $\lambda_i\geqq 0$), so that

$$\text{(1)} \quad \frac{d(|y|^2)}{dt}\leqq\left(\sum_{i=1}^n\lambda_i\right)\sum_{i=1}^n|z_i|^2=\operatorname{trace}(A+A^*)|y|^2$$
$$=(\operatorname{tr}A+\operatorname{tr}A^*)|y|^2=2\operatorname{Re}[\operatorname{tr}A]|y|^2=2X(t)|y|^2.$$

Let y be a nontrivial solution of $dy/dt=Ay$, so that $|y|^2$ is never 0. Hence we

get from (1),

$$\frac{1}{|y^2|}\frac{d|y|^2}{dt} \leqq 2X(t). \tag{2}$$

Integrating (2) from 0 to t gives

$$\log \frac{|y(t)|^2}{|y(0)|^2} \leqq 2\int_0^t X(t)dt$$

or

$$|y(t)|^2 \leqq |y(0)|^2 \exp 2\int_0^t X(t)dt \leqq |y(0)|^2 \exp 2 \limsup_{t\to\infty} \int_0^t X(t)dt,$$

which implies that $y(t)$ is bounded over $[0, \infty)$ provided hypothesis (ii) holds.

Also solved by the proposer.

Number of Solutions of a Congruence System

4746 [1957, 437]. *Proposed by Leonard Carlitz, Duke University*

Let p be an odd prime. Show that the number of solutions of the system

$$x + y + z \equiv 3, \qquad xyz \equiv 1 \pmod{p} \tag{1}$$

is equal to $p-2-(-3/p)$, where $(-3/p)$ denotes the quadratic character of -3. If $p>3$, show that the number of solutions is the same for the system

$$x + y + z \equiv 3, \qquad x^3 + y^3 + z^3 \equiv 3 \pmod{p}. \tag{2}$$

Solution by W. J. Blundon, Memorial University of Newfoundland. We have

$$(x - y)^2 = (x + y)^2 - 4xy \equiv (3 - z)^2 - 4/z \equiv (z - 1)^2(z - 4)/z \pmod{p}.$$

If $x \equiv y \pmod{p}$, then $z \equiv 1$ or 4. This leads to exactly two solutions, namely (1, 1, 1) and $((p-1)/2, (p-1)/2, 4)$.

If $x \not\equiv y \pmod{p}$, then $(z-4)/z$ is a quadratic residue modulo p. Let $(z-4)/z \equiv n^2$, where $n^2 \not\equiv 0, 1, -3 \pmod{p}$. Thus z assumes $\frac{1}{2}(p-1)-2$ or $\frac{1}{2}(p-1)-1$ distinct values according as -3 is or is not a quadratic residue modulo p. It follows that the number of solutions with $x \not\equiv y$ is double this number, namely $p-4-(-3/p)$. The total number of solutions of this system is thus $p-2-(-3/p)$.

If we apply the transformation $2x = Y+Z$, $2y = Z+X$, $2z = X+Y$ to the system (1) we obtain

$$\begin{aligned} X + Y + Z &\equiv x + y + z \equiv 3 \pmod{p}, \\ X^3 + Y^3 + Z^3 &\equiv (X + Y + Z)^3 - 3(Y + Z)(Z + X)(X + Y) \\ &\equiv (x + y + z)^3 - 24xyz \equiv 3 \pmod{p}. \end{aligned}$$

Since the transformation is linear with nonzero determinant, the number of

solutions is the same for both systems. The condition $p>3$ excludes the exceptional case $(X, Y, Z)\equiv(0, 0, 0)$ which is a solution of the second system and yet is the transform of $(x, y, z)\equiv(0, 0, 0)$ which is not a solution of the first.

It is interesting to note that, upon replacing congruence modulo p by equality in the above, we may use the substitution $(z-4)/z=m^2/n^2$ to obtain an infinity of rational solutions for the Diophantine equation

$$X^3 + Y^3 + Z^3 = 3,$$

and hence an infinity of integral solutions of

$$a^3 + b^3 + c^3 = 3d^3,$$

namely

$$a = 4n^3 - 3mn^2 - m^3, \qquad b = 4n^3 + 3mn^2 + m^3,$$
$$c = -5n^3 - 3m^2n, \qquad d = n^3 - m^2n.$$

Also solved by Emil Grosswald, Emma Lehmer, D. C. B. Marsh, and the proposer. Late solution by Yoshio Matsuoka.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The World of Mathematics. Edited with commentaries and notes by James R. Newman. Simon and Schuster, New York, 1956. 4 volumes. 2536 pp. $20.00.

In recent months it has not been possible for the literate American to remain unaware of this anthology of James Newman's favorite essays on mathematical and scientific topics which has reached the book dividend stage as a gift offering with membership in the Library of Science. Mr. Newman's desire to share these literary treasures with the general reader deserves applause. For no more convenient, no more delightful, no less painful means of acquainting onself with some of the basic ideas of the mathematical-scientific elements in our culture is to be found in our scientific literature.

The collection will yield but little gain to the reader who seeks exclusively the exposition of specific mathematical techniques or the reprinting of the great landmarks in the development of the subject since few technical papers of renowned mathematicians or scientists have been included. The emphasis is on the *opinions* of other renowned mathematicians, scientists, logicians, philosophers and historians of science, and on the importance of the great mathematical and scientific landmarks. The anthology can be of great value to one who not only believes with Norman Campbell that "the only way to understand what

Einstein did is to look at the symbols in which his theory must ultimately be expressed and to realize that it was reasons of symbolic form, and such reasons alone, which led him to arrange the symbols in the way he did and no other" (p. 1829), but who also wishes to become acquainted with authoritative critical thought on the foundations, development, and philosophy of mathematics, on the applicability of its methods and conclusions to problems in the social, physical, and natural sciences, and on its essential character as a creative art.

Inasmuch as Mr. Newman has drawn on masses of diverse material to accomplish his several purposes, the cover-to-cover reader will be aware of repetitive themes that weave, each in its own way, threads of continuity throughout the anthology. Mr. Newman has done little in his editing to bind the whole together in terms of any one theme, but the reader who is biased in his preference for subject matter will quite naturally use that interest as the element of linkage.

The reader, for example, who shares the author's intellectual sensitivity to the intrinsic beauty of mathematical form, will highlight John Sullivan's statement that "The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds. . . . We are the law-givers of the universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the material universe itself" (p. 2021). He will relish Sylvester's oft-quoted words: "May not Music be described as the Mathematic of sense, Mathematic as Music of the reason?" (p. 364). He will find pleasure in the opening sentence of Section 10 in Hardy's famous essay: "A mathematician, like a painter or a poet, is a maker of patterns" (p. 2027).

An anthologist must be granted the freedom to select the material that he feels is pertinent to his purpose and to ride his "prejudices," as Mr. Newman might express it. This anthologist has wished to include "material to suit every taste and capacity" and has offered enough material in each category to leave an indelible impression.

The general reader would have benefited still more by the inclusion of editorial criticism in the editorial commentaries which have been interspersed throughout the four volumes. These "settings" are mostly descriptive and biographical in character and often become little gems of characterization. But controversial viewpoints quite naturally appear in the essays and some guidance in evaluating the differing opinions seems necessary for the untutored reader.

There remain a few general observations. The dozens of quotations introducing the essays have been chosen for their aptness with great skill. The text seems singularly free of typographical errors. The two columns in the description of the Egyptian problems on p. 177 seem to be arranged in reverse order.

There is so much of general value in this "small library of the literature of mathematics"—there are 133 items in the collection—that it cannot be disregarded by the mathematics teacher on any level. Students in the various scien-

tific disciplines have already been dipping into it as an easily accessible and rewarding source of reference material. Perhaps the collection was not originally compiled "with the hope of encouraging the study of science" as Campbell declares of his own work on p. 1796, but it may well come to serve that worthy purpose. Reflecting colorful shafts of light from the mathematical firmament, *The World of Mathematics* is recommended for the pleasure and information it has to offer and should be shelved within easy reaching distance of every mathematical work-desk.

CAROLYN EISELE
Hunter College

The Computing Laboratory in the University. Edited by Preston C. Hammer. University of Wisconsin Press, Madison, 1957. xv+236 pp. $6.50.

This book is a compilation of papers presented at a conference with the same title as the book, held at Madison in August, 1955. The scope of the papers is very broad. Some discuss the philosophical and sociological impact of large, high-speed computers on society. Others give surveys of computing applications in specific scientific and engineering areas, such as weather prediction, fluid dynamics, chemistry, physics, and in the aircraft industry. Most of the papers are on the subjects of equipping, organizing, and financing a university computing laboratory, curricula in computing, and the acute shortage of personnel, unanimously agreed to get worse, in computational mathematics. The number of universities with modern computing equipment has more than doubled in the two years since this conference. It is unfortunate that these valuable papers were not available earlier, but since the same problems of organization, whether to build, buy, borrow or rent, and how to raise money without running a service bureau, are facing dozens of schools today, the book is still timely. These papers, by distinguished administrators and scientists in universities and in industry, should be read by most academic administrators as background for deciding how their schools can use the expensive, but probably the most versatile research tool yet devised—the digital computer.

WILLIAM VIAVANT
University of Oklahoma

BRIEF MENTION

Numerical Calculus. By William Edmund Milne. Princeton University Press, 1954. x+393 pp. $4.50.

The current interest in computing machines, numerical analysis, approximations, interpolation, finite differences, numerical integration, and curve fitting is sufficient to merit calling the attention of our readers to a book on the above subjects which is noted for its clarity, even though that book was first published in 1949.

Mathematics of Investment. By William L. Hart. Heath, New York, 1958. viii+494 pp. $6.75.

The fourth edition of Hart's well-known *Mathematics of Investment* needs no review other than to announce its availability in a highly legible new format. The book is designed to "meet the needs and the ability of the typical student in a college of business administration."

Calculus. By Edward S. Smith, Meyer Salkover and Howard K. Justice. Wiley, New York, 1958. xiii+520 pp. $6.50.

Another old friend in a new edition. The inclusion of chapters in solid analytic geometry, and elementary differential equations will be welcomed by many potential users.

From Zero to Infinity. By Constance Reid. Crowell, New York, 1958. 145 pp. $3.00.

Not a book on number theory, but a delightfully written light little book on the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. It can be read by anyone and enjoyed by most.

Van Nostrand's Scientific Encyclopedia (3rd Ed.). Van Nostrand, New York, 1958. vii+1839 pp. $30.00.

This beautifully illustrated one-volume encyclopedia contains a surprising amount of knowledge. Specialists may be somewhat disappointed to note that the discussion of topology is restricted to rubber sheet geometry rather than point set topology. One may question the desirability of including formulas for differentiation under the integral sign in a special paragraph of the same title without stating the conditions which must be satisfied. Still, it is picayune to criticize. A specialist should not expect a complete treatment of his pet subject in a one-volume encyclopedia. This welcome volume is superior to anything similar that this reviewer has seen.

How to Study. By Clifford T. Morgan and James Deese. McGraw-Hill, New York, 1957. vii+136 pp. $1.50.

Teachers may wish to call the attention of their students to this interesting little book. It contains many excellent hints and suggestions on the gentle art of study. Unfortunately, the section devoted to mathematics is one of the poorest in the book. The examples seem to have been taken largely from the sub-high school level rather than the college level. Nevertheless, teachers of mathematics may wish to recommend it to their students in the hope that the general organization which it suggests may help the student arrange his schedule so that there is sufficient time to study mathematics; and to suggest, also, the excellent "How to Study, How to Solve" by Dadourian (Addison-Wesley) for helpful hints on actual mathematical problems.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1958.

Columbia University, July 7 to August 15: Professor Stein, introduction to linear algebra, fundamental concepts of mathematics; Professor Mendelson, differential equations; Mr. Gordon, probability; Professor Griffin, functions of a complex variable; Dr. Taft, higher algebra; Professor Taylor, general topology, theory of group representations.

Fordham University. July 7 to August 14: Summer Institute for high school teachers of mathematics offering courses in probability and statistical inference and basic concepts and structures of geometry; in the Graduate School, differential equations, integral equations, and foundations of analysis.

Massachusetts Institute of Technology. July 8 to July 18: Probability theory and its application to operations research, under the direction of Professor George P. Wadsworth assisted by Dr. James K. Knowles with lectures by Philip M. Morse, Professor of Physics, and Dr. Herbert P. Galliher, Operations Research Project. July 21 to August 1: Mathematical analysis and simulation in industrial operations, sponsored by the School of Industrial Management with the cooperation of the Operations Research Project and other departments, under the direction of Dr. Herbert P. Galliher, Operations Research Project; other participants include George E. Kimball, Visiting Professor of Chemistry, Edward H. Bowman, Robert B. Fetter, Dean Arden, Gregory Chow, William S. Jewell, Richard B. Maffei. August 4 to August 15: Design and analysis of scientific experiments, Professor Harold Freeman and Dr. Marian Krzyzaniak.

St. John's University, June 30 to August 8: Professor Tolle, foundations of mathematics, solid analytic geometry, material and methods of modern mathematics.

University of California, Berkeley, June 16 to July 26: Visiting Professor Besicovitch, almost periodic functions. June 16 to July 26 and July 28 to September 6: Professors Neyman, Fix, and Hodges, research seminar in statistical problems of health; Dr. David (University College, London), research seminar in statistical studies of structural relations in physical sciences; Professor Blackwell, individual research leading to higher degrees.

University of Minnesota, Institute of Technology, June 16 to July 19: Professor Wilcox, vector analysis; Professor Pope, numerical analysis, introduction to programming modern digital calculators; Visiting Professor Roy, elements of statistical inference—theory, elements of statistical inference—applications. July 21 to August 23: Professor Stenberg, advanced calculus.

A COMPUTING CENTER AT SOUTHERN METHODIST UNIVERSITY

Southern Methodist University announces the opening of a Computing Laboratory on its campus. A new building houses the Univac Scientific 1103 Computer, the Remington Rand Service Bureau and the SMU Computing Laboratory offices and classrooms. The computer is operated jointly by Remington Rand as a service to industry and by SMU as an academic service for research and teaching. The SMU operation is associated

with the University's new Graduate Research Center. Professors and students have free use of the machine for academic research and training in computer work. Training programs are available for faculty and students. Computing projects are now under way in the fields of engineering, mathematics, psychology, law, religion, management and others. SMU will make the computer available to other universities and nonprofit institutions on a cooperative arrangement involving only a nominal fee for overhead, and invites inquiries leading to such use of the machine. SMU regards its laboratory as a regional university computing facility.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The following are additions to the list (this MONTHLY, vol. 65, 1958, p. 303) of Summer Institutes for high school teachers:

Fordham University. The institute for high school teachers, announced on page 378 of this issue, is sponsored by the National Science Foundation. Alternates selected by other NSF Summer Institutes may apply by May 24, 1958.

University of Missouri, School of Mines and Metallurgy, Rolla, June 9–August 1: H. Q. Fuller, Director.

RUSSIAN TECHNICAL JOURNAL TRANSLATION

A leading Russian technical journal will soon be translated into English on a regular basis by the American Society of Mechanical Engineers. Under a $35,000 grant from the National Science Foundation, ASME will publish the bimonthly "Journal of Applied Mathematics and Mechanics." The Society has undertaken the task of translation "in an attempt to correct the present situation in which the Russians are familiar with the content of most, if not all, of our technical publications, while only a few of theirs are translated for use by the English-speaking world."

The magazine contains the latest theoretical and practical advances made by Russian scientists in mathematics, fluid dynamics and solid state physics. Copies will be sold, on a subscription basis, to any interested persons or groups at an annual rate of $35 for the six issues. Subscriptions may be ordered from the Order Department, The American Society of Mechanical Engineers, 29 West 39th Street, New York 18, New York. Publication begins with the first 1958 issue.

Professor George Herrmann of Columbia University will serve as editor of the translated journal.

PRELIMINARY ACTUARIAL EXAMINATIONS

For a number of years Educational Testing Service, the organization that assists the Society of Actuaries in the preparation of the Preliminary Actuarial Examinations, has participated in a program of Graduate Record Examinations given at a large number of schools in the United States and Canada. The tests are used as a measure of student achievement and ability to do graduate work, and are currently required of applicants for National Science Foundation fellowships.

The Graduate Record Examinations consist of an Aptitude Test that includes a verbal section similar to the Society's Part 1 and several Advanced Tests, including one in mathematics similar to Part 2. They are scheduled four times a year, in January, April, July, and November. Detailed information regarding the nature of these examinations, the centers where they are given, *etc.*, may be obtained by writing Educational Testing Service, 20 Nassau Street, Princeton, New Jersey, for a copy of the Bulletin of Information for Candidates.

The Board of Governors of the Society of Actuaries has approved the recommendation of the Education and Examination Committee that credit for Part 1 of the Preliminary Actuarial Examinations be allowed to candidates who take the Graduate Record Aptitude Test and who achieve a score on the verbal section determined by the Educa-

tion and Examination Committee to be equivalent to the Part 1 passing score. Similarly, credit for Part 2 will be allowed to candidates who take the Graduate Record Advanced Test in Mathematics. In either case, to be eligible for credit a candidate must have taken the Graduate Record Examinations while an undergraduate or graduate student in a college or university. An application to the Society for credit may be completed either in advance of taking the Graduate Record Examinations, or within two years of having taken them as an undergraduate or graduate student.

In the future, the Preliminary Actuarial Examination pass list published by the Society will include the names of students who have received credit for Graduate Record Examinations. They will also be listed appropriately in the Year Book of the Society.

PERSONAL ITEMS

Professor S. W. McCuskey of Case Institute served as the representative of the Association at a meeting of the Council on Documentation Research held at Western Reserve University on February 3 and 4, 1958.

Professor Abe Gelbart of Syracuse University has been appointed Editor of *Scripta Mathematica* and Director of Yeshiva University's Institute of Mathematics. He will succeed the late Professor Jekuthiel Ginsburg and will assume his new duties July 1, 1958.

Associate Professor A. R. Harvey of San Diego State College has received a Fulbright award to lecture at the College of Arts and Sciences in Baghdad, Iraq.

Mr. R. C. Jurgensen, Chairman of the Mathematics Department at Culver Military Academy, has been appointed to hold the first endowed chair in mathematics in the school's history.

Professor Emeritus Solomon Lefschetz of Princeton University has been elected a corresponding Member of the Academie des Sciences of Paris.

Catholic University of America: Assistant Professor R. W. Moller has been appointed Associate Professor; Dr. P. P. Saworotnow has been appointed Assistant Professor.

Lebanon Valley College: Mr. G. F. Heck has been elected the first president of the newly-organized Davis Chapter of the Industrial Mathematics Society. Other officers elected were: Mr. Earl Edris, vice president; Mr. E. A. Anderson, secretary; Mr. John Ray, treasurer; and Professor R. J. Wagner, faculty advisor.

Lincoln Memorial University sponsored its First Annual Symposium for the Improvement of Science and Mathematics Teachers of Secondary Schools on October 18 and 19, 1957. The program included an invited address by Professor F. L. Wren, George Peabody College, on *The teaching of mathematics in secondary schools.* The symposium will be repeated in October, 1958.

University of New Hampshire: Dr. S. L. Ross and Mr. F. J. Lorenzen, Jr. have been appointed Assistant Professors.

Mr. W. R. Abel, University of Nebraska, has been appointed Assistant Professor.

Dr. Brian Abrahamson of the University of Cape Town has been appointed Professor at Rhodes University.

Assistant Professor J. J. Andrews, St. Louis University has been appointed Associate Professor.

Professor M. G. Boyce, Vanderbilt University, on leave this semester, is a Visiting Fellow at Princeton University.

Mr. D. A. Breault, Sylvania Electric Products Co., Waltham, Mass., is on leave of absence with the U. S. Army Signal Corps.

Mr. H. H. Brown, Ramo-Wooldridge Corporation, has accepted a position as research mathematician for the Missile Systems Division of the Lockheed Aircraft Corporation, Palo Alto, California.

Miss Winifred K. Burroughs, Ohio Wesleyan University, has been appointed Instructor.

Dr. D. L. Clark, Oregon State College, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, New York, N. Y.

Mr. J. G. Cox, formerly Head, Math. Services Dept., VITRO Corp., Eglin Air Force Base, Florida has been appointed Associate Professor at Alabama Polytechnic Institute.

Assistant Professor H. F. Cullen, University of Massachusetts, has been appointed Associate Professor.

Assistant Professor M. R. Demers, University of Nevada, has been appointed Associate Professor.

Professor Benjamin Epstein of Wayne State University is on leave and has been appointed Visiting Professor at Stanford University.

Dr. J. D. Esary of the University of California, Berkeley, has accepted a position as research engineer for the Boeing Airplane Company, Seattle, Washington.

Dr. Jean B. Feidner, University of Buffalo, has accepted an appointment as Assistant Professor at Hobart and William Smith Colleges.

Professor R. N. Festa of the Ministry of Education, Vienna, Austria, is on leave and has been appointed Visiting Professor at the State College of Washington.

Professor A. H. Fox, Union College, has been appointed Chairman, Department of Mathematics.

Professor J. E. Freund, Virginia Polytechnic Institute, has been appointed Professor at Arizona State College, Tempe.

Dr. R. D. Glauz of the General Electric Company has accepted a position as technical specialist for Aerojet-General Corporation, Sacramento, California.

Dr. Carl Hammer of Remington-Rand International Corporation has accepted a position as engineering specialist for Sylvania Electric Products, Inc., Waltham, Massachusetts.

Assistant Professor Aaron Herschfeld of Canisius College has been appointed Assistant Professor at Pennsylvania State University.

Dr. B. M. Ingersoll, City College of New York, has been appointed Assistant Professor at San Diego State College.

Professor J. B. Jackson, University of South Carolina, has been appointed Visiting Professor at Mary Washington College.

Assistant Professor H. T. Jones, Emmanuel Missionary College, has been appointed Associate Professor.

Assistant Professor R. J. Kasriel, Georgia Institute of Technology, has been appointed Associate Professor.

Assistant Professor L. H. Lange of Valparaiso University, as the recipient of a Danforth Foundation Teacher Study Grant, continues on leave at the University of Notre Dame.

Rev. C. J. Lewis, Fordham University, has been appointed Assistant Professor.

Assistant Professor R. W. MacDowell, University of Rochester, has been appointed Associate Professor at Antioch College.

Mrs. Ruth J. MacKichan, University of North Dakota, has been appointed Assistant Professor.

Dr. M. E. Mahowald, formerly with the General Electric Co., Cincinnati, Ohio, has been appointed Assistant Professor at Xavier University.

Colonel J. D. Matheson of the United States Army has been appointed a senior research analyst for Melpar, Inc., Falls Church, Virginia.

Mr. H. W. McCurdy, Graduate Student, Florida State University, has been appointed Instructor at Bradley University.

Dr. M. A. Medick, City College, has accepted a position as senior staff scientist at the Research and Advanced Development Division of AVCO Manufacturing Corporation, Lawrence, Massachusetts.

Assistant Professor Paul Meier, Johns Hopkins University, has been appointed Associate Professor at the University of Chicago.

Mr. V. H. Morrill, Tarleton State College, has been appointed Assistant Professor.

Dr. J. A. Nickel, Willamette University, has been appointed Assistant Professor.

Mr. Enuenwemba Obi, teaching fellow at the University of Kansas, has been appointed part-time Instructor at Park College.

Dr. J. B. O'Toole of Hughes Aircraft Company has accepted a position as senior engineer for National Cash Register Company, Hawthorne, California.

Dr. P. A. Penzo, University of Pittsburgh, has accepted a position as senior research engineer for CONVAIR Astronautics, San Diego, California.

Assistant Professor W. J. Pervin, University of Pittsburgh, has been appointed Assistant Professor at Pennsylvania State University.

Mr. C. F. Pinzka, Xavier University, has been appointed Instructor at the University of Cincinnati.

Assistant Professor L. E. Pursell, Grinnell College, has been appointed Associate Professor.

Associate Professor Gustave Rabson, Antioch College, has accepted a position as senior mathematician in the Research Center of American Optical Company, Southbridge, Massachusetts.

Assistant Professor J. B. Roberts, Reed College, has been appointed Associate Professor.

Associate Professor F. V. Rohde, University of Florida, has been appointed Professor at the University of Chattanooga.

Dr. D. W. Sasser, Yale University, has accepted a position as staff member of Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Seymour Schuster, Polytechnic Institute of Brooklyn, has been appointed Associate Professor.

Associate Professor L. L. Scott, University of Mississippi, has been appointed Associate Professor at Southwestern at Memphis.

Professor M. E. Shanks, Purdue University, is on leave and is at the Institute for Advanced Study.

Dr. Abraham Spitzbart of the General Electric Company, Cincinnati, Ohio has been appointed Associate Professor at the University of Wisconsin in Milwaukee.

Dr. T. D. Sterling, University of Michigan Engineering Research Institute, has been appointed Assistant Professor at Michigan State University.

Associate Professor E. A. Sturley of Allegheny College has been appointed Associate Professor at the University of Southern Illinois.

Assistant Professor Patrick Suppes, Stanford University, has been appointed Associate Professor.

Mr. R. E. Thomas of North American Aviation, Inc., has accepted a position as assistant division consultant for Batelle Memorial Institute, Columbus, Ohio.

Associate Professor Henry Van Engen of Iowa State Teachers College has been appointed Professor at the University of Wisconsin.

Professor C. P. Wells, Michigan State University, is on leave and has been appointed a research associate at the California Institute of Technology.

Dr. F. J. Weyl, formerly Director of the Mathematical Science Division, Office of Naval Research, has been appointed Director, Naval Analysis Group, Office of Naval Research.

Professor S. S. Wilks, Princeton University, has been appointed Chairman of the Mathematics Division of the National Academy of Sciences-National Research Council.

Professor J. C. Wilson of Fenn College has been awarded a Danforth Teacher Study Grant for 1958–59 and will study toward his Ph.D. in mathematics at Case Institute of Technology.

Dr. E. M. Zaustinsky, University of Southern California, has been appointed Assistant Professor at San Jose State College.

Professor Israel Abrams, Drexel Institute of Technology, died January 4, 1958.

Mr. Nathan Barotz, Brooklyn College, died on August 16, 1957.

Professor Emeritus R. M. Deming, retired, Upper Iowa University, died January 13, 1958. He was a charter member of the Association.

Professor Carl M. Erikson, Eastern Michigan College, died on February 10, 1958. He had been a member of the Association for thirty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1957

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1957. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

There was a surplus of $1,876 in the Current Fund for 1957. The balances in the regular funds of the Association are less because of the decrease in the value of the securities owned by the Association and because of printing expenses in some cases.

Assets of the Association	January 1, 1957	December 31, 1957
M & T Trust Co., Buffalo	30,857.02	25,703.17
Savings Accounts	19,297.28	75,999.51
Securities	111,828.64	100,974.12
	$161,982.94	$202,676.80
Funds of the Association		
Current Fund	$ 450.03	$ 325.80
Carus Fund	23,763.26	20,235.25
Chace Fund	12,369.13	8,808.28
Houck Fund	12,082.88	11,309.92
Chauvenet Fund	1,420.06	1,328.43
Dunkel Fund	19,917.92	15,889.55
General Fund	40,518.72	38,414.63
	$110,522.00	$ 93,311.66
Visiting Lecturers Fund	$ 42,942.44	$ 48,424.53
Fund for Committee on Undergraduate Program	8,069.77	55,669.95
Fund for Committee on Films	448.73	85.66
Fund for Committee on High School Contests	—	2,184.80
	$161,982.94	$202,676.80

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 175 persons have been elected to membership by the Board of Governors on applications duly certified.

IAIN T. ADAMSON, Ph.D. (Princeton) Lecturer, Queen's University of Belfast, Northern Ireland.

JULIA E. ADKINS, Ph.D. (Ohio State) Asst. Professor, Central Michigan College.

RALPH J. ANDREE, B.S. (Oklahoma S.U.) Student, Oklahoma State University.

MARY F. BAEURLE, M.A. (Columbia) Teacher, Central High School, Paterson, N. J.

ALEXANDER R. BEDNAREK, B.S. (New York S.T.C.) Teaching Fellow, University of Buffalo.

JOHN J. BENEDETTO, Student, Boston College.

RONALD J. BENICE, Student, University of Buffalo.

RICHARD J. BETTS, Student, Rice Institute.

CLAIR W. BLACK, Ed.D. (Columbia) Dean, School of Eng. & Sc., Fairleigh Dickinson University.

CLIFFORD D. BLACK, B.S. (Jacksonville S. C.) Teaching Fellow, University of Kentucky.

THOMAS BONNER, M.S. (Illinois) Grad. Asst., University of Illinois.

MRS. FLORENCE W. BORGESON, B.S. (Delaware) Teacher, Roosevelt Junior High, Westfield, N. J.

DUSHAN BOYANOVITCH, M.E. (Stevens) Grad. Asst., Stevens Institute of Technology.

MILTON BROOKS, M.A. (Pennsylvania) Head, Dept. of Math., West Philadelphia High School.

MICHAEL BUCKLEY, JR., M.S. (Purdue) Asst. Professor, Santa Clara University.

THOMAS W. CAIRNS, M.S. (Oklahoma S.U.) Grad. Asst., Oklahoma State University.

ALESSANDRO C. CALIANESE, B.S. (St. Peter's) Grad. Stud., University of Notre Dame.

JAYME M. CARDOSO, D.Sc. (Universidade do Parana) Professor, Universidade do Parana, Brazil.

MRS. BEVERLY N. CARRELL, M.A. (George Washington) Teacher, Woodrow Wilson High School, Washington, D. C.

PHILIP W. CARRUTH, Ph.D. (Illinois) Asso. Professor, Swarthmore College.

ALAN P. CLARK, M.A. (Columbia) Teacher, Parsippany High School, Parsippany, N. J.

ANDREW CONTI, JR., Student, Youngstown University.

HENRY C. COOKE, M.S. (N. Carolina S. C.) Asst. Professor, North Carolina State College.

DONALD T. COTTINGHAM, B.S. (Harvard) Teaching Asst., University of Washington.

THADDEUS F. CROMWICK, B.S. (Worcester Poly. Inst.) Vice President, Telemat Corp., Worcester, Mass.

DAVID R. CROSBY, M.S. (Harvard) Elec. Engr., Radio Corp. of America, Camden, N. J.

DONALD I. CUTLER, B.S. (Illinois) Asst. Math., System Development Corp., Santa Monica, Calif.

DAVID F. DAWSON, Ph.D. (Texas) Asst. Professor, University of Missouri.

JAMES E. DELANY, Student, San Diego State College.

WILLIAM H. DENT, JR., B.A. (Maryville Coll.) Teaching Fellow, University of Kentucky.

MRS. RUTH E. DILLAVOU, M.A. (Columbia) Instr., Gonzaga University.

ROBERT G. DOUGLAS, Student, University of Kansas.

KENNETH B. DUNHAM, Student, Georgia Institute of Technology.

ORVILLE E. ETTER, M.S. (Kansas S.T.C.) Asst. Professor, Fort Hays Kansas State College.

ARMSTRONG O. FARMER, B.S. (Lafayette Coll.) Draftsman, American Bridge Co., Trenton, N. J.

FAY FARNUM, Ph.D. (Cornell) Asst. Professor, University of Arizona.

RAY H. FARRIS, M.Ed. (Abilene) Asst. Professor, York College, Nebraska.

ROBERT C. FISHER, Ph.D. (Kansas) Asst. Professor, Ohio State University.

JOE W. FITZPATRICK, M.A. (Texas) Instr., Trinity University.

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MARILYN J. WOODYARD, M.S. (Northwestern) Programmer, IBM Research Lab., Poughkeepsie, N. Y.

GAIL S. YOUNG, Ph.D. (Texas) Professor, University of Michigan.

THE OCTOBER MEETING OF THE INDIANA SECTION

A joint meeting of the Indiana Section of the Mathematical Association of America and the Mathematics Division of the Indiana Academy of Science was held at DePauw University, Greencastle, Indiana, on October 18, 1957. Professor I. W. Burr, Chairman of the Mathematics Division of the Academy, presided.

There were 69 in attendance including 37 members of the Association.

The following papers were presented:

1. *The mathematics of the future*, by Professor P. D. Edwards and Professor C. F. Brumfiel, Ball State Teachers College.

Professor Edwards stressed the increasing importance of having high school teachers of mathematics who are much more thoroughly trained in advanced mathematics than was considered necessary a relatively few years ago. This applies not only to the traditional topics needed by the engineer and the physical scientist but also to the needs of workers in other fields. It was pointed out that very definite improvements in the teachers' preparation may be made by drastic changes in the content of high school mathematics. Professor Brumfiel made a progress report on an experimental program now being supervised by Ball State which was made possible by a grant from the National Science Foundation.

2. *A characterization of n-adic equivalence relations*, by Professor J. L. Lawrence, Wabash College and International Business Machines, introduced by the Secretary.

The concepts of symmetry, transitivity, composition, and equivalence associated with dyadic relations were generalized to apply to the n-adic case. In the generalized scheme, symmetry and transitivity persist as necessary and sufficient conditions for an n-adic relation to be an equivalence relation. Furthermore, as in the dyadic case, an n-adic relation is transitive if and only if the relation contains the composition of the relation with itself.

3. *The concept of surface integral*, by Mr. L. H. Turner, Purdue University, introduced by the Secretary.

A continuous parametric surface (T, A) is a continuous mapping T from a subset A of E_2 into three space E_3, $p=T(w)$, $w=(u, v)\in A$, $p=(x, y, z)\in E_3$. The usual definition of area in terms of the Jacobians of the mapping is inadequate in modern analysis. An adequate definition was given by Lebesgue in terms of sequences of polyhedral mappings which approach the original mapping. When the area is finite, four measures ϕ, V_1, V_2, V_3 may be defined by means of these sequences on a certain ring of Borel subsets of A such that V_1, V_2, and V_3 are absolutely continuous with respect to ϕ. The Radon-Nikodym derivatives $\theta_1(w)=dV_1/d\phi$, $\theta_2(w)=dV_2/d\phi$, $\theta_3(w)=dV_3/d\phi$ exist and satisfy $\theta_1^2+\theta_2^2+\theta_3^2=1$ a.e. (ϕ). The vector $\theta(w)=(\theta_1, \theta_2, \theta_3)$ may be thought of as the directional normal to the surface (T, A) at $T(w)$. Then if $D=\{(\theta_1, \theta_2, \theta_3):\theta_1^2+\theta_2^2+\theta_3^2=1\}$ and $f(x, y, z, \theta_1, \theta_2, \theta_3)$ is any function which is Borel measurable and bounded on $T(A)\times D$, the integral $H(T, A, f)=(A)\int\int f(T(w), \theta(w))d\phi$ exists and is the integral of f over (T, A).

4. *Periodic solutions of nonlinear differential equations*, by Professor W. R. Fuller, Purdue University.

In this paper, which was expository in nature, were indicated some types of nonlinear differential equations and systems of such equations, for which existence theorems for periodic solutions have been studied. This includes systems containing a small parameter, ϵ, which for $\epsilon=0$ have periodic solutions. In particular the study of systems of the form $x+\sigma^2x=\epsilon f(x, t; \epsilon)$ where σ^2x and f are n-vectors has been very fruitfully attacked by a method of L. Cesari which has been applied to a wide class of problems by Cesari, Hale, Gambill, Bailey and the author (see, *e.g.*, Atti Acad. Italia, vol. 11, 1940, pp. 633–692; Bull. AMS 60, 1954, pp. 64–66, 367; 62, 1956, p. 567; 63, 1957, p. 271).

5. *Localization experiment for teaching geometry*, by Professor A. D. Hummel, Ball State Teachers College, introduced by the Secretary.

A method of determining the x, y, and z coordinates of points within a body by means of X-rays was reviewed. Projected images of scales are used as measuring sticks in two radiographs. The source of the X-rays has different positions for the two radiographs. Elementary geometry is used to compute the coordinates of any point appearing in both. In the experiment described for teaching, a small source of visible light is substituted for the X-ray source. Data obtained from shadows is used to calculate the length of the object. Verification of the result by direct measurement should strengthen the students' faith in geometry.

6. *Undergraduate curricula—some brave experiments and cogent lessons*, by Professor A. E. Ross, University of Notre Dame.

There are many critical problems which confront one in the task of implementing an effective undergraduate program in mathematics. Today such a program must not only appeal to the people who have traditionally relied upon mathematical tools, but it must also appeal to the users of new mathematics, most of whom come from the humanities, from commerce, and from the "preprofessional" groups. The novelty of the new undergraduate curriculum lies not only in the new content but also in the growing recognition that mathematical manipulative skills alone do not develop the capacity for intelligent application nor do these skills alone justify considering mathematics as one of the liberal arts. To design a proper blending of the new and the old, of ideals and of skills, and to make this blend accessible to the very young brings us close to the fundamental questions in the art of communication. The purpose of this talk was to discuss and to illustrate some of these questions.

7. *A graphical solution for a particular finite series*, by Dr. R. H. L. Howe, Eli Lilly and Company, Lafayette, Indiana, introduced by the Secretary.

There are problems in engineering and the physical sciences which require the evaluation of y given as a finite series of the form $y=1/x_1+\cdots+1/x_n$, where $x_1, \cdots, x_n$ are positive or negative real numbers. Let $y=1/R$. Then R can be found graphically using a simple geometric principle. When a large number of terms is involved, this graphical method is particularly time saving. It is thus very useful in checking problems such as those concerning resistance of resistors in parallel, capacity of condensers in series, focal length in optical systems, and total resistance or conduction coefficient of materials in heat transfer and transmission.

8. *On the inter-relationship of applications and mathematical research*, by Dr. K. L. Nielsen, U. S. Naval Avionics Facility, Indianapolis, Indiana.

Emphasizing the constant increase and breadth of scope in the utilization of mathematics in contemporary nonmathematical fields, the author concentrated primarily on the inter-relationship between mathematical research and some technological developments, electronic calculating machines, and the philosophy of education. He discussed the role of the mathematician in industry, industry's utilization of mathematics, the development of new mathematics, and the need for closer cooperation between the educators and those engaged in research and the application of mathematics.

J. C. POLLEY, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The second annual meeting of the New Jersey Section of the Mathematical Association of America was held at Fairleigh Dickinson University, Rutherford, New Jersey, on November 2, 1957. Dean A. E. Meder Jr., retiring Chairman of the Section, presided at the morning session; Dean C. W. Black, Fairleigh Dickinson University, presided at the luncheon and during the address by Dr. Morris Meister; and Professor B. E. Meserve, newly-elected Chairman of the Section, presided at the afternoon session. 104 persons registered.

At the business meeting, Professor B. E. Meserve of the State Teachers College at Upper Montclair was elected Chairman of the Section for 1957–58. Dr. J. D. Daugherty, Eastside High School, Paterson, was elected a member of the Executive Committee to serve for three years. Mr. R. S. Lockhart, Chairman of the Contest Committee, reported that the Committee recommended that: (a) the New Jersey Section endorse the national contest as a means of promoting a maximum of local participation (in secondary school mathematics contests); (b) the national contest results be used as a guide in selecting participants to enter contests such as the New Jersey State Mathematics Day at Rutgers; and (c) the Chairman of the Section appoint a committee to administer the national contest in New Jersey. These recommendations were adopted.

The following papers were presented, all by invitation of the Program Committee:

1. *The theory of braids*, by Professor Emil Artin, Princeton University.

The meaning of the postulates of group theory can be easily explained by the example of the group of all braids with a given number of strings. The composition consists in tying one braid to another, the unit element is the braid with unentangled strings, and the inverse is a certain reflection of a given braid. The word problem solves the topological question of classification of braids.

2. *Numerical calculation of flows in rivers and reservoirs*, by Professor J. J. Stoker, Institute of Mathematical Sciences, New York University, introduced by the Secretary.

The development of calculating machines and methods of numerical analysis make it possible to compute flows in rivers over hundreds of miles for periods of weeks. The present paper reports the successful outcomes of such calculations in three cases: 1) 375 miles of the Ohio River; 2) the junction of the Ohio and the Mississippi; and 3) Kentucky Reservoir at the mouth of the Tennessee River.

3. *A new program in the teaching of physics*, by Dr. Morris Meister, Bronx High School of Science.

The course in physics for secondary schools being developed by the Physical Science Study Committee at MIT stresses major developments of physics as a logical and integrated whole. Designed to meet the needs of the top twenty-five percent of students, it aims to give them intellectual and cultural insight into present-day human activity and achievement. Teaching aids, including a textbook, laboratory manual, teacher's manual, films and film strips, monographs related to the course, demonstrations, and laboratory experiments and apparatus kits are in process of development. The course is now being tried out in seven schools in various parts of the country. Next summer there will be at least four institutes for physics teachers interested in the new program.

4. *The work of the Commission on Mathematics, College Entrance Examination Board*, by Dean A. E. Meder, Jr., Rutgers University.

The Commission seeks the modernization, modification, and improvement of the college preparatory secondary curriculum so that it may be oriented to the needs of mathematical, physical, and social science, and of industry and technology as these needs exist in the second half of the twentieth century. The basis for change is to be found in the point of view of modern mathematics, which looks for patterns, not tricks (Sawyer). Specific suggestions for changes in school algebra, geometry, and trigonometry were given, as well as proposals for elective work in elementary analysis, statistical inference, and other fields.

I. L. BATTIN, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The third annual meeting of the Northeastern Section of the Mathematical Association of America was held at Dartmouth College, Hanover, New Hampshire, on Novem-

ber 30, 1957, with Professor Stanley Bezuszka, S.J., of Boston College, Chairman of the Section, presiding. There were 75 people in attendance, including 59 members of the Association.

A brief welcoming address was delivered by President John Dickey of Dartmouth College. At the business meeting, the following officers were elected: Chairman, Professor D. E. Richmond of Williams College; Vice-Chairman, Professor N. H. McCoy of Smith College; Secretary-Treasurer, Professor Anne F. O'Neill of Wheaton College.

Professor Richmond presented the report of the Committee that had been appointed to consider the matter of Section participation in the National Contest for High Schools. It was voted to refrain from participation in the National Contest for the next year. It was also voted to appoint a committee to study the details of setting up the contest for the year after this, and to investigate alternatives to participation in the contest as well as the work done in other countries in contests of this kind.

The following program was presented:

1. *In retrospect*, by President C. V. Newsom, New York University.

Dr. Newsom opened his remarks by pointing out the fact that in his experience as a mathematician and educator he had been embarrassed frequently by his inability to describe in somewhat popular fashion the precise role of the mathematician in the total field of knowledge. Such an explanation is essential, for example, in any attempt to justify the inclusion of mathematics in a general education. Then he indicated his concurrence with many of the viewpoints expressed by John von Neumann, in the article entitled "The Mathematician," published in the newsletter of the Canadian Mathematical Congress, sixth issue, May, 1957. In particular, he commented upon von Neumann's observation that, "The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any science which interprets experience on a higher than purely descriptive level." Dr. Newsom stated that the theoretical scientist is studying a nature that is basically inscrutable, but he has been successful in developing patterns of many types that correlate known data. Any such pattern becomes of great utility if it serves to cover other phenomena which were not considered or even not known at the time when the pattern was evolved. He said that the most sophisticated of all such patterns is the collection of propositions, called axioms, which characterize a situation under consideration. Such a scheme possesses values with which mathematicians generally are familiar. If the role of the mathematician is described essentially as that of pattern-maker, he takes on a stature that is comprehensible to nonscientists as well as scientists. The speaker culminated his remarks by indicating that the adoption of such a viewpoint would have a desirable effect upon the curriculum, and it might serve to accomplish the very desirable end of drawing pure and applied mathematicians more closely together.

2. *Statistical number geometry*, Professor H. J. Zassenhaus, McGill University.

Pack as many as possible convex bodies that are congruent under translation to a given convex body into a given Jordan region. This basic problem of statistical number geometry, a new mathematical discipline, was solved in two dimensions by N. Oler recently, proving a conjecture of Zassenhaus. Thus a new proof of C. A. Rogers' 1951 result that the critical lattices of Minkowski's geometry of number in the plane form the densest pointsets admissible for any given Minkowski-distance was obtained. Generalizations to other dimensions by Zassenhaus and distance functions by N. Smith and M. Rahman were also discussed.

3. *Operational meaning in mathematics*, by Professor Reinhard Korgen, Bowdoin College.

Professor Korgen discussed the problem of keeping mathematical machinery from obscuring operational meaning in the applications of mathematics. Thus an "average" may be defined as the number G which leaves invariant the value of some prescribed function of a set of measurements when G is substituted for each and every one; the context of a given problem will determine what

function to prescribe as the one to be left unchanged in value. The theory of measurement itself was involved in other examples adduced: it was pointed out that certain numerical laws in the natural sciences seem to report on nature when they are used instead to introduce scale-form in the measurement of quantities for which scale-form has not otherwise been shaped.

4. *Mechanics for undergraduates*, by Professor Garrett Birkoff, Harvard University.

Professor Birkoff described an ideal course in mechanics for undergraduates.

5. *University of Illinois School mathematics program*, by Professor Max Beberman, University of Illinois.

The purpose of the University of Illinois Committee on School Mathematics is to create a four-year program in college preparatory mathematics which treats mathematics in the manner of contemporary mathematicians and which stimulates interest among young people in the continued study of mathematics. In carrying out this purpose, the UICSM has written texts for students and teachers. The topics covered in these texts include the distinction between numbers and numerals; algebraic manipulations based on arithmetic generalizations [commutativity, *etc.*]; graphing equations and inequalities; a postulational development of euclidean geometry in which geometric objects are taken to be sets of points; the idea of a deductive theory in which models are constructed for various sets of postulates; mathematical induction; induction proofs of the laws of exponents; complex numbers; integral rational functions; polynomial equations; circular functions.

ANNE F. O'NEILL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, St. John's University, Collegeville, May 17, 1958.

MISSOURI, University of Missouri, Columbia, May 3, 1958.

NEBRASKA

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN

SOUTHERN CALIFORNIA

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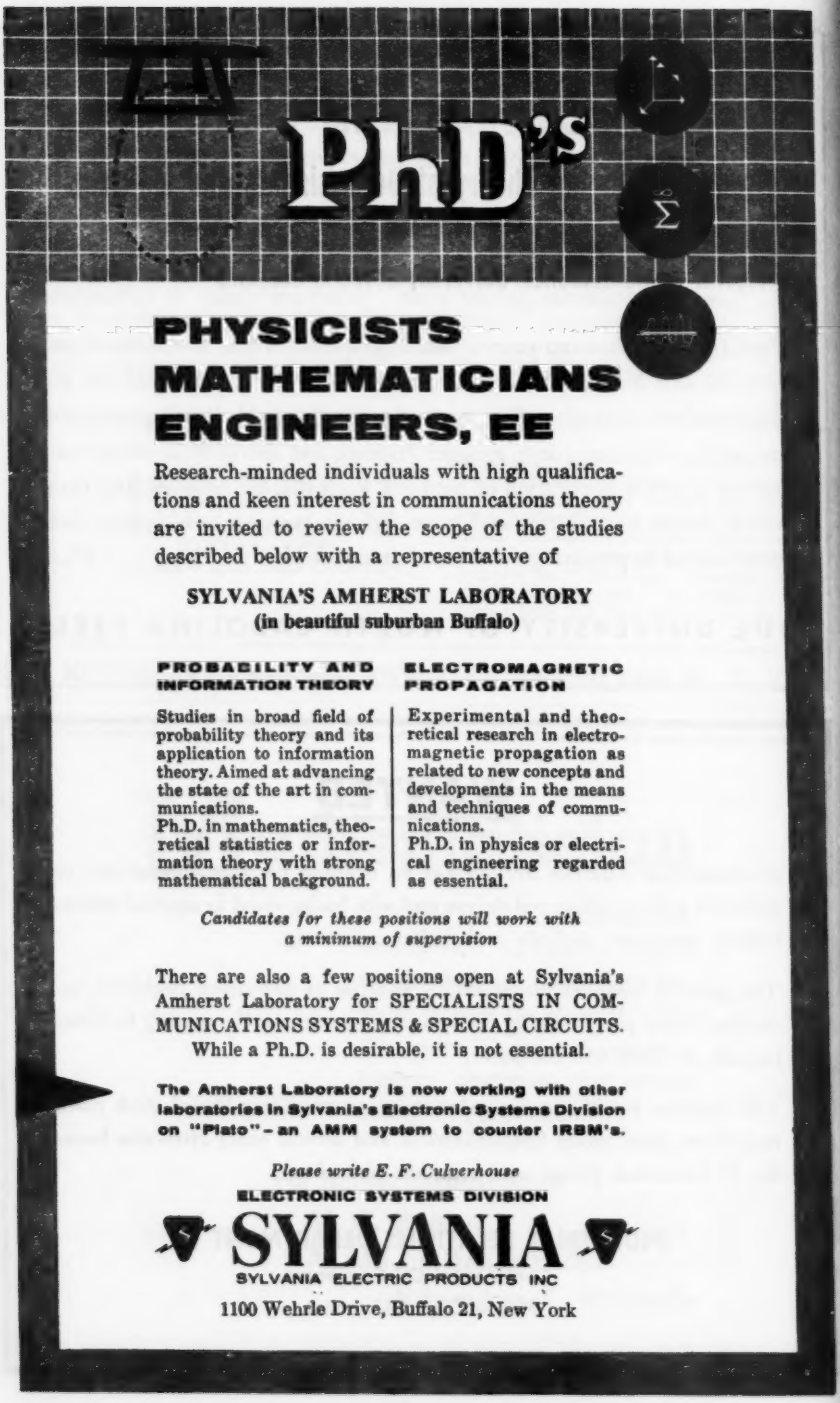

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